# On the range of the transient frog model on $\mathbb{Z}$

Arka Ghosh[*]        Steven Noren[†]        Alexander Roitershtein[‡]

February 9, 2015; Revised September 15, 2016

### Abstract

We observe the frog model, an infinite system of interacting random walks, on $\mathbb{Z}$ with an asymmetric underlying random walk. For certain initial frog distributions we construct an explicit formula for the moments of the leftmost visited site, as well as their asymptotic scaling limits as the drift of the underlying random walk vanishes. We also provide conditions in which the lower bound can be scaled to converge in probability to the degenerate distribution at 1 as the drift vanishes.

## 1   Introduction

Consider the following interacting random walks model on $\mathbb{Z}$: initially at each site $x$ there is a fixed number $\eta_x$ of sleeping particles ("frogs"), and there is a certain number ($\eta_0$) of active frogs at the origin. The active frogs perform in discrete time, simultaneously and independently of each other, a biased (say, to the right) nearest-neighbor random walk on $\mathbb{Z}$. When an active frog visits a site $x$, it activates the $\eta_x$ sleeping frogs at $x$, in which each active frog performs the same underlying random walk starting from its initial location, all random walk transitions being independent of each other. The active frogs continue to visit other sleeping frogs and activate them. This model for an infinite number of interacting random walkers is called the frog model on $\mathbb{Z}$ (with drift).

A frog model is called *recurrent* if 0 is visited infinitely often by active frogs w.p.1, and *transient* if 0 is visited only finitely often w.p.1. It is shown in [12] that the zero-one dichotomy actually takes place, namely a one-dimensional frog model is either transient or recurrent. Both necessary and sufficient conditions for recurrence of the frog model on $\mathbb{Z}$ based on the configuration of frogs and the drift of the random walk are provided in [12]. Recurrence for variants of the frog model on more general graphs have been first explored in [26] (for the symmetric random walk on $\mathbb{Z}^d$) and subsequently in [3], [22] and [23]. Shape theorems for the model in $\mathbb{Z}^d$ have been obtained in [1, 2]. For further background on the

---

[*]Department of Statistics, Iowa State University, Ames, IA 50011, USA; e-mail: apghosh@iastate.edu
[†]Department of Mathematics, Iowa State University, Ames, IA 50011, USA; e-mail: srnoren@iastate.edu
[‡]Department of Mathematics, Iowa State University, Ames, IA 50011, USA; e-mail: roiterst@iastate.edu

frog model and its variants, refer to [23]. For an account of the most recent activity in the area see [5, 8, 15, 16]. In particular, [8] generalizes a recurrence criterion of [12] to a model in $\mathbb{Z}^d$, [15] and [16] provide recurrence and transience criteria for the frog model on trees, and [5] studies survival of particles in a one-dimensional variation of the model, also partially extending some of the results of [12].

The frog model can be interpreted as an information spreading network [2, 23]. The underlying idea is that an active frog holds some information and shares it with sleeping frogs when they meet, activating the sleeping frogs who then spread the information along their random walk path. A closely related to our model *particle process on* $\mathbb{Z}$, describing the evolution of a virus in an infinite population (e.g., computer network), has been considered in [18, 19] and [5]. The model is also a discrete-time relative of the one-dimensional *stochastic combustion process* studied in [7, 24].

In this article, we will explore the behavior of the frog model, in particular its range, when transience is assumed. We specifically introduce a drift component to the random walk and explore how its magnitude affects the range of visited sites in the model. Each active frog will move one integer to the right with probability $p \in (\frac{1}{2}, 1)$, or one integer to the left with probability $1 - p$. Thus the underlying random walk is transient to the right. We define the drift constant $\rho := \frac{1-p}{p} \in (0, 1)$. The drift term $\rho$ can be seen as a measure of "transience" of this frog model; small values of $\rho$ indicate more frequent rightward movement by the frogs, whereas values of $\rho$ close to 1 more closely resemble recurrence with a slight rightward drift (see, for instance, formula (5) below for a concrete random walk result). Of particular interest is the collective behavior of the frogs as $\rho \uparrow 1$. By Theorem 2.1 in [12], this frog model is transient when we assume an identical distribution of frogs on the nonnegative sites. In particular, w.p.1 there must be only a finite number of visited sites to the left of the origin. That is, in the language of [5], transience implies *local extinction* for our model.

In this article, while assuming $\eta_x = 0$ for all $x < 0$, we will first explore the single-frog case, i.e., the frog model in which $\eta_x = 1$ for all $x \geq 0$ (cf. [5, 18, 19]). We will provide exact and asymptotic results for the moments of the lower bound of the range, which will be used in convergence theorems. After that, we will move to more general choices for $\eta$ and show that, under certain conditions, the frog model's lower bound will behave asymptotically similar to that of the single-frog case. Finally, we will provide asymptotic bounds for moments of the frog model range when $\eta$ is supported on all of $\mathbb{Z}$.

The rest of the paper is organized as follows. A short Section 2 introduces notations and certain technical tools necessary for our proofs. The three subsequent sections constitute the main body of the article. The single-frog case is considered in Section 3. A class of more general initial configurations of frogs $\eta$ is discussed in Section 4. The consideration of configurations supported on $\mathbb{Z}$ is discussed in Section 5.

# 2    Preliminaries

For calculations of characteristics of the random variable representing the range of the frog model, we make use of common notation in analytic number theory and combinatorics. For $a \in \mathbb{R}$, $c \in \mathbb{N} \cup \{\infty\}$, $|q| < 1$, the *q-Pochhammer symbol* is defined by $(a; q)_c := \prod_{j=0}^{c-1}(1 - aq^j)$. The following equality can be verified directly from the definition: $(q^{x+1}; q)_\infty = \frac{(q;q)_\infty}{(q;q)_x}$, $x \in \mathbb{N}$.

The $q$-Pochhammer symbol is one of the key functions in the construction of $q$-analogs in number theory, and is often used in the theory of basic hypergeometric functions and analytic combinatorics [4, 10, 13]. For example, [13] provides the following identities:

$$(z;q)_\infty = \sum_{n=0}^\infty \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n} z^n \qquad \text{and} \qquad \frac{1}{(z;q)_\infty} = \sum_{n=0}^\infty \frac{z^n}{(q;q)_n}. \tag{1}$$

Thus for a nonzero $q \in (-1, 1)$, $(z;q)_\infty$ and $\frac{1}{(z;q)_\infty}$ are both analytic functions of $z$ on $(0, 1)$. The $q$-gamma function is defined as $\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}$. The $q$-digamma function $\psi_\rho$ is defined as $\psi_q(z) = \frac{1}{\Gamma_q(z)} \frac{\partial \Gamma_q(z)}{\partial z} = -\ln(1-q) + \ln q \sum_{n=0}^\infty \frac{q^{n+z}}{1-q^{n+z}}$.

To facilitate our calculation of the range moments, we need to develop notation for Bell polynomials. The Bell polynomials are defined (see, for instance, [6, 21]) as the triangular array of polynomials $B_{m,k}$, $m \geq k$, given by

$$B_{m,k}(x_1, \ldots, x_{m-k+1}) = \sum \frac{m!}{k_1! \cdots k_{m-k+1}!} \left(\frac{x_1}{1!}\right)^{k_1} \cdots \left(\frac{x_{m-k+1}}{(m-k+1)!}\right)^{k_{m-k+1}}, \tag{2}$$

where the sum is taken over all sequences of nonnegative integers $\{k_1, \ldots, k_{m-k+1}\}$ satisfying

$$k_1 + k_2 + \cdots + k_{m-k+1} = k \quad \text{and} \quad k_1 + 2k_2 + \cdots + (m-k+1)k_{m-k+1} = m. \tag{3}$$

The $m^{th}$ complete Bell polynomial is $B_m(x_1, \ldots, x_m) = \sum_{k=1}^m B_{m,k}(x_1, \ldots, x_{m-k+1})$. While the Bell polynomials has many intriguing details that can be explored in combinatorial number theory (see, for instance, [6] and references therein), we are mostly concerned with their presence in the celebrated Faà di Bruno's formula [17, 20] for derivatives of composite functions:

$$\left(\frac{d}{dt}\right)^m f\big(g(t)\big) = \sum_{k=1}^m f^{(k)}\big(g(t)\big) \, B_{m,k}\big(g'(t), g''(t), \ldots, g^{(m-k+1)}(t)\big). \tag{4}$$

Throughout this paper $f(x) \sim g(x)$ as $x \to c$ stands for $\lim_{x \to c} \frac{f(x)}{g(x)} = 1$.

## 3 The single frog per site case

We will first assume that $\eta_x = 1$ for each nonnegative integer $x$ and $\eta_x = 0$ elsewhere. Let $\rho = \frac{1-p}{p} \in (0, 1)$ and consider the corresponding frog model. Let $W_\rho$ represent the random variable for the negative of the minimum of the visited sites in this model. For convenience, we will construct a family of mutually independent random variables $(X_\rho)_{\rho \in (0,1)}$ that all share the same probability space $(\Omega, \mathcal{F}, P)$ such that $X_\rho$ and $W_\rho$ share the same distribution.

The distribution function of $X_\rho$ can be easily found by observing that, by the rightward tendency of the initial active frog, all frogs w.p.1 will eventually be woken. Furthermore, if $(S_n)_{n \geq 0}$ has the distribution of the underlying random walk, then

$$P(S_n = 0 \text{ for some } n \geq 1 \,|\, S_0 = 1) = \rho. \tag{5}$$

With this observation, we see that for all $x \geq 0$,

$$P(X_\rho \leq x) = P(X_\rho < x + 1) = \prod_{j=1}^{\infty}(1 - \rho^{x+j}) = (\rho^{x+1}; \rho)_\infty. \tag{6}$$

It is a simple exercise to see that $P(X_\rho = 0) = \prod_{j=1}^{\infty}(1 - \rho^j) = (\rho; \rho)_\infty$. For all $x > 0$, the value of $X_\rho$'s probability density function is

$$P(X_\rho = x) = P(X_\rho \leq x) - P(X_\rho \leq x - 1) = \prod_{j=1}^{\infty}(1 - \rho^{x+j}) - \prod_{j=0}^{\infty}(1 - \rho^{x+j})$$

$$= (1 - (1 - \rho^x)) \prod_{j=1}^{\infty}(1 - \rho^{x+j}) = \rho^x (\rho^{x+1}; \rho)_\infty. \tag{7}$$

With the density known, we would now like to study the behavior of $X_\rho$ for values of $\rho$ close to 1, where the frog model more closely resembles the recurrent case. Objects that describe the concentration of the distribution of $X_\rho$ include the central statistics of the random variable, such as the mode and the expectation. For fixed $\rho$, we define a unique representative of the mode statistic, named $M_\rho$, by

$$M_\rho := \min\{x \geq 0 : P(X_\rho = x) \geq P(X_\rho = n) \text{ for all } n \geq 0\}.$$

While the mode statistic is not usually observed compared to other central statistics of a class of random variables, it is a quick calculation that can often provide insight on asymptotic. Also, as we observe later, the concentration of $X_\rho$ around its mode will be influential to its limiting behavior. With that in mind, we present the following result:

**Theorem 3.1.** *For any $\rho \in (0,1)$,* $\left\lfloor \dfrac{\ln(1 - \rho) - \ln\rho}{\ln\rho} \right\rfloor \leq M_\rho \leq \left\lfloor \dfrac{\ln(1 - \rho) - \ln(2 - \rho)}{\ln\rho} \right\rfloor$,

*where $\lfloor a \rfloor$ is the largest integer less than or equal to a. In particular, $M_\rho \sim \frac{\ln(1-\rho)}{\ln\rho}$ as $\rho \to 1$.*

Note that for all $\rho \in (0,1)$, the difference between the two bounds in the theorem's conclusion always belongs to the open interval $(0, 2)$. Hence, even for values of $\rho$ that make the bounds extremely large, the theorem narrows down $M_\rho$ to two possibilities.

*Proof of Theorem 3.1.* Considering $x$ to be a continuous variable, we note that

$$\frac{d}{dx}(\rho^{x+1}; \rho)_\infty = \frac{d}{dx}\exp\left(\sum_{j=1}^{\infty}\ln(1 - \rho^{x+j})\right) = \left(\frac{d}{dx}\sum_{j=1}^{\infty}\ln(1 - \rho^{x+j})\right)(\rho^{x+1}; \rho)_\infty, \tag{8}$$

It is enough to confirm that where $||f||_\infty = \sup\{|f(x)| : x \in [a, b]\}$. Thus the series $\sum_{j=1}^{\infty}||\frac{\rho^{x+j}}{1-\rho^{x+j}}||_\infty$ absolutely converges for any closed interval $[a, b]$ we choose. Hence, the series in (8) is differentiable, and furthermore, by [25, Theorem 7.17], the derivative of the series is the series of the derivatives. Thus $\frac{d}{dx}(\rho^{x+1}; \rho)_\infty = -\ln(\rho)(\rho^{x+1}; \rho)_\infty \sum_{j=1}^{\infty}\frac{\rho^{x+j}}{1-\rho^{x+j}}$. Using the above derivative, we can find the derivative of the density function for $X_\rho$:

$$\frac{d}{dx}\left\{(\rho^{x+1}; \rho)_\infty \rho^x\right\} = \frac{d}{dx}\left\{\prod_{j=1}^{\infty}(1 - \rho^{x+j})\rho^x\right\} = \ln(\rho)\rho^x(\rho^{x+1}, \rho)_\infty\left(1 - \sum_{j=1}^{\infty}\frac{\rho^{x+j}}{1 - \rho^{x+j}}\right).$$

4

Notice that $\ln(\rho)\rho^x(\rho^{x+1},\rho)_\infty$ is nonzero for all $x > 0$. Therefore, by the 1$^{\text{st}}$-derivative test for critical points, $M_\rho$ is an integer within 1 away from the positive value $m$ such that $\sum_{j=1}^\infty \frac{\rho^{m+j}}{1-\rho^{m+j}} = 1$. Since $\rho^{x+j} \le \frac{\rho^{x+j}}{1-\rho^{x+j}} \le \frac{\rho^{x+j}}{1-\rho^{x+1}}$ for all $x > 0$ and all $j \in \mathbb{N}$, we obtain $\frac{\rho^{m+1}}{1-\rho} = \sum_{j=1}^\infty \rho^{m+j} \le \sum_{j=1}^\infty \frac{\rho^{m+j}}{1-\rho^{m+j}} = 1 \le \sum_{j=1}^\infty \frac{\rho^{m+j}}{1-\rho^{m+1}} = \frac{1}{1-\rho^{m+1}}\frac{\rho^{m+1}}{1-\rho}$. The result follows accounting for the fact that $M_\rho$ is integer-valued. $\qquad\square$

We would now like to compare the mode of $X_\rho$ with the moments of the random variable. Calculating the moments directly from definition can be quite a challenge, given the convoluted expression of the density in (14). Therefore, we take the more circuitous option of first finding the cumulants of $X_\rho$. With $M_{X_\rho}(t) := E(e^{tX_\rho})$, the *cumulant generating function* $g_\rho(t)$ of $X_\rho$ is defined as $g_\rho(t) := \log\big(M_{X_\rho}(t)\big)$. We then define the $m^{th}$ *cumulant* $\kappa_\rho^{(m)}$ of $X_\rho$ to be the $m^{\text{th}}$ derivative of the cumulant generating function evaluated at 0: $\kappa_\rho^{(m)} := g_\rho^{(m)}(0)$. Cumulants can be used to determine moments through the use of Faà di Bruno's formula (4) (see, for instance, [20, 21] and references therein), which is the direction we will take. It turns out that the cumulants of $X_\rho$, though unable to be written down using fundamental functions, can be expressed as straight-forward series representations.

**Lemma 3.2.** *The cumulant generating function of $X_\rho$ is $g_\rho(t) = \sum_{k=1}^\infty \ln\left(\frac{1-\rho^k}{1-e^t\rho^k}\right)$. Furthermore, for each $m \in \mathbb{N}$, the $m^{th}$-cumulant of $X_\rho$ is $\kappa_\rho^{(m)} = \sum_{k=1}^\infty \frac{k^{m-1}\rho^k}{1-\rho^k}$.*

*Proof.* Using (1), we calculate the moment generating function of $X_\rho$:

$$M_{X_\rho}(t) := E(e^{tX_\rho}) = (\rho;\rho)_\infty \sum_{x=0}^\infty \frac{e^{tx}\rho^x}{(\rho;\rho)_x} = \frac{(\rho;\rho)_\infty}{(e^t\rho;\rho)_\infty} = \prod_{k=1}^\infty \frac{1-\rho^k}{1-e^t\rho^k}. \tag{9}$$

Taking the natural logarithm of (9) gives us the desired formula for $g_\rho(t)$. To find $\kappa_\rho^{(m)}$, we first find the $m^{\text{th}}$ derivative of $g_\rho(t)$:

$$g_\rho^{(m)}(t) = \sum_{k=1}^\infty \left(\frac{d}{dt}\right)^m \ln\left(\frac{1-\rho^k}{1-e^t\rho^k}\right) = \sum_{k=1}^\infty \left(\frac{d}{dt}\right)^m \left(\ln(1-\rho^k) + \sum_{j=1}^\infty \frac{(e^t\rho^k)^j}{j}\right)$$

$$= \sum_{k=1}^\infty \sum_{j=1}^\infty j^{m-1}(e^t\rho^k)^j = \sum_{j=1}^\infty \sum_{k=1}^\infty j^{m-1}(e^t\rho^k)^j = \sum_{j=1}^\infty \frac{j^{m-1}e^{tj}\rho^j}{1-\rho^j}.$$

Moving the derivative inside of the summation is justified by Theorem 7.17 in [25], at least for $t \in [-\delta,\delta]$ for small enough $\delta > 0$. Setting $t = 0$ gives us the result for $\kappa_\rho^{(m)}$. $\qquad\square$

Finding the moments of a random variable through its cumulants is a well-known technique (see, for instance, [20, 21] and references therein), but we show the details here for completeness. Applying Faà di Bruno's formula (4) to $M_{X_\rho}(t) = e^{g_\rho(t)}$ shows that

$$M_{X_\rho}^{(m)}(t) = e^{g_\rho(t)}\sum_{k=1}^m B_{m,k}(g_\rho'(t),\ldots,g_\rho^{(m-k+1)}(t)) = M_{X_\rho}(t)B_m(g_\rho'(t),\ldots,g_\rho^{(m)}(t)).$$

Setting $t = 0$, we arrive at the following result.

5

**Theorem 3.3.** *Using the notation from Section 1 and Lemma 3.2, for each $\rho \in (0,1)$ and for all $m \in \mathbb{N}$,*

$$E(X_\rho^m) = B_m\left(\kappa_\rho^{(1)}, \kappa_\rho^{(2)}, \cdots, \kappa_\rho^{(m)}\right). \tag{10}$$

**Corollary 3.4.** $E(X_\rho) = \sum_{x=1}^{\infty} \frac{\rho^x}{1-\rho^x} = \frac{\psi_\rho(1) + \ln(1-\rho)}{\ln \rho}$ *and* $\mathrm{Var}(X_\rho) = \sum_{x=1}^{\infty} \frac{x\rho^x}{1-\rho^x} = \frac{\psi_\rho'(1)}{\ln^2 \rho}$, *where $\psi_\rho$ is the q-digamma function defined in Section 2.*

The exact calculation of the moments in Theorem 3.3 can be quite unwieldy for large values of $m$. Thus, it's insightful to observe simpler asymptotic formulas instead. In [11], the following were proven for the input series of the Bell polynomials in 3.3 as $\rho \uparrow 1$ :

$$\sum_{k=1}^{\infty} \frac{\rho^k}{1-\rho^k} \sim \frac{1}{1-\rho} \ln \frac{1}{1-\rho} \sim \frac{\ln(1-\rho)}{\ln \rho} \tag{11}$$

$$\sum_{k=1}^{\infty} \frac{k^j \rho^k}{1-\rho^k} \sim \frac{j!\zeta(j+1)}{(1-\rho)^{j+1}} \sim \frac{j!\zeta(j+1)}{-\ln^{j+1}\rho}, \qquad j \geq 1, \tag{12}$$

where $\zeta(j) = \sum_{k=1}^{\infty} 1/k^j$ is the Riemann zeta function.

Notice that for each $m \in \mathbb{N}$, the polynomial $B_m(x_1, \ldots, x_m)$ includes a term of the form $x_1^m$, and this term has coefficient 1. Furthermore, all other terms of $B_m(x_1, \ldots, x_m)$ are a multiple of some variable other than $x_1$ with a positive coefficient. Indeed, the polynomials $B_{m,k}$ possess this property by virtue of (2) and (3), because the latter ensures that $k_2 = \cdots = k_{m-k+1} = 0$ and hence $k_1 = k = m$. Clearly, $B_m$ inherits the feature from $B_{m,k}$'s.

Note that from (11), $\left(\frac{\ln \rho}{\ln(1-\rho)}\right)^m \left(\kappa_\rho^{(1)}\right)^m \to 1$ as $\rho \uparrow 1$. Also, it follows from (12), that for all $j \in \{2, 3, 4, \ldots, m\}$, $\left(\frac{\ln \rho}{\ln(1-\rho)}\right)^j \kappa_\rho^{(j)} \sim \frac{(j-1)!\zeta(j)}{-\ln^j(1-\rho)} \to 0$ as $\rho \uparrow 1$. Since every term in (10) is a multiple of some cumulant other than $\kappa_\rho^{(1)}$, except for the $(\kappa_\rho^{(1)})^m$ term, multiplying $E(X_\rho^m)$ by $\left(\frac{\ln \rho}{\ln(1-\rho)}\right)^m$ and taking the limit as $\rho \uparrow 1$ eliminates all terms except for the aforementioned term which converges to 1. Therefore, we can obtain the following asymptotic result:

**Corollary 3.5.** *For all $m \in \mathbb{N}$, $E(X_\rho^m) \sim \left(\frac{\ln(1-\rho)}{\ln \rho}\right)^m$ as $\rho \uparrow 1$.*

To further clarify the special role of the number $z_\rho = \frac{\ln(1-\rho)}{\ln \rho}$ in our model consider $U_\rho := \#\{\text{frogs who reached the site } -z_\rho\}$, where, for simplicity and clarity of the subsequent computation, we treat $z_\rho$ as an integer. Then, using (5) and the Markov property, observe that $E(U_\rho) = \sum_{x=0}^{\infty} P(S_n = -z_\rho \text{ for some } n \geq 1 \mid S_0 = x) = \sum_{x=0}^{\infty} \rho^{x+z_\rho} = \frac{\rho^{z_\rho}}{1-\rho} = 1$. This result can be heuristically interpreted as an illustration of the fact that $-z_\rho$ serves as the most distant place, though barely, is still accessible to the frog population (in average only one frog can reach that far).

We now seek to determine the asymptotic of the distribution of $X_\rho$ when $\rho \uparrow 1$. Theorem 3.1 and Corollary 3.4 both hint that $X_\rho$ grows at roughly the same rate as $\frac{\ln(1-\rho)}{\ln \rho}$ as $\rho$ rises close to 1. The nature of this growth can be revealed by the asymptotic of the variance, which grows sufficiently slow to guarantee scaling limits for $X_\rho$.

**Theorem 3.6.** *Let $Y_\rho$ share the same distribution as $\frac{\ln \rho}{\ln(1-\rho)} X_\rho$ for each $\rho \in (0,1)$. Then, as $\rho \uparrow 1$, $Y_\rho \to 1$ in probability. That is, for all $\epsilon > 0$, $\lim_{\rho \to 1^-} P(|Y_\rho - 1| > \epsilon) = 0$.*

*Proof.* By Corollary 3.4 and (11), $E(Y_\rho) = \frac{\ln \rho}{\ln(1-\rho)} E(X_\rho) \to 1$. To show the convergence in probability, it is enough to prove that $\text{Var}(Y_\rho) \to 0$ as $\rho \uparrow 1$. By Corollary 3.4 and (12), $\text{Var}(X_\rho) \sim \frac{1!\zeta(2)}{(1-\rho)^2} \sim \frac{\pi^2}{6} \frac{1}{\ln^2 \rho}$. Thus, $\text{Var}(Y_\rho) = \frac{\ln^2 \rho}{\ln^2(1-\rho)} \text{Var}(X_\rho) \sim \frac{\pi^2}{6} \frac{1}{\ln^2(1-\rho)} \to 0$. Since $E(Y_\rho) \to 1$ and $\text{Var}(Y_\rho) \to 0$, convergence in probability follows. $\square$

Aside from the probabilistic implications of Theorem 3.6, we can also use the generating functions of $Y_\rho$ to construct some limit identities involving the $q$-Pochhammer symbol. Since $Y_\rho \to 1$ in probability (and thus in distribution), the moment generating function of $Y_\rho$ converges pointwise to that of the degenerate variable at 1. Similarly, the characteristic function also converges pointwise to the characteristic function of the same degenerate variable. This observation leads to the following corollary:

**Corollary 3.7.** *For $z > 0$,* $\lim_{\rho \uparrow 1} \dfrac{(\rho; \rho)_\infty}{\left( z^{\frac{\ln(\rho)}{\ln(1-\rho)}} \rho; \rho \right)_\infty} = z$. *For $t \in \mathbb{R}$,* $\lim_{\rho \uparrow 1} \dfrac{(\rho; \rho)_\infty}{\left( e^{it \frac{\ln(\rho)}{\ln(1-\rho)}} \rho; \rho \right)_\infty} = e^{it}$.

We wish to prove a stronger convergence of the sequence of random variables $\{Y_\rho\}_{\rho \in (0,1)}$. Theorem 3.6 implies that an appropriate discretization $\{Y_{\rho_n}\}_{n \in \mathbb{N}}$ of $\{Y_\rho\}_{\rho \in (0,1)}$ can be chosen to achieve the almost sure convergence. The following result identifies a class of sequences $\{\rho_n\}_{n \in \mathbb{N}}$ that ensures the almost sure convergence of the discrete sequence $Y_{\rho_n}$.

**Proposition 3.8.** *Let $\{\rho_n\}_{n \in \mathbb{N}} \subseteq (0,1)$ be a sequence such that $\rho_n \uparrow 1$ and $\frac{1}{\ln(1-\rho_n)} \in \ell^2$. Let $\{Y_{\rho_n}\}_{n \in \mathbb{N}}$ be a sequence of random variables in the probability space $(\Omega, \mathcal{F}, P)$ such that $Y_{\rho_n}$ has the same distribution as $\frac{\ln \rho_n}{\ln(1-\rho_n)} X_{\rho_n}$ for each $n$. Then, as $n \to \infty$, $Y_{\rho_n} \to 1$ a. s.*

*Proof.* Let $\epsilon > 0$ be given. By the Borel-Cantelli Lemma, a sufficient condition for a. s. convergence is that $P(|Y_{\rho_n} - 1| > \epsilon) \in \ell^1$. Choose $\rho \in (0,1)$ such that $|E(Y_\rho) - 1| < \frac{\epsilon}{2}$. Then, using Chebyshev's Inequality,

$$P(|Y_\rho - 1| > \epsilon) \leq P(|Y_\rho - E(Y_\rho)| + |E(Y_\rho) - 1| > \epsilon)$$

$$\leq P\left( |Y_\rho - E(Y_\rho)| > \frac{\epsilon}{2} \text{ or } |E(Y_\rho) - 1| > \frac{\epsilon}{2} \right) = P\left( |Y_\rho - E(Y_\rho)| > \frac{\epsilon}{2} \right) \leq \frac{4}{\epsilon^2} \text{Var}(Y_\rho).$$

As noted above, $\text{Var}(Y_\rho) = \frac{\ln^2 \rho}{\ln^2(1-\rho)} \text{Var}(X_\rho) \sim \frac{\pi^2}{6} \frac{1}{\ln^2(1-\rho)}$. Replacing $\rho$ with the terms of $\{\rho_n\}_{n \in \mathbb{N}}$, we see that $P(|Y_{\rho_n} - 1| > \epsilon) \in \ell^1$, proving our result. $\square$

The class of sequences defined in the hypothesis of the proposition above includes those of the form $\rho_n = 1 - e^{-n^c}$, where $c > \frac{1}{2}$ is constant. For further research, we wish to broaden the class of sequences that lead to the a. s. convergence. For instance, instead of assuming that the models corresponding to different values of $\rho$ are independent, one can consider a standard hierarchial coupling of the underlying random walks leading to the setting where $P(X_{\rho_1} \leq X_{\rho_2}) = 1$ if $\rho_1 < \rho_2$. In that case one can consider for example $\rho_n = 1 - n^{-c}$, $c > \frac{1}{2}$, and imitating Etemadi's proof of the law of large numbers (see, for instance, Section 2.4 in [9]), namely first considering subsequences $k(n) = \lfloor \alpha^n \rfloor$ with an arbitrary $\alpha > 1$ and then using the fact that the ratio of $\frac{\ln \rho_{k(n)}}{\ln(1-\rho_{k(n)})}$ and $\frac{\ln \rho_{k(n+1)}}{\ln(1-\rho_{k(n+1)})}$ converges to $\alpha^c$ when $n \to \infty$, prove the almost sure convergence of $Y_{\rho_n}$ for the sequence $\rho_n = 1 - n^{-c}$ by finally taking $\alpha$ to 1.

# 4 More general frog distributions $\eta$

Now let's consider the frog model with drift in which its frog distribution $\eta = \{\eta_x\}_{x=0}^{\infty}$ is a sequence of natural numbers, i.e., $\eta_x \geq 1$ for all $x \geq 0$ and $\eta_x = 0$ elsewhere. Main results of this section are stated in Theorems 4.1 and 4.2 below.

According to Theorem 2.1 in [12], in order to have transience in the frog model with drift $\rho$, and hence an almost surely finite minimum of its range, we must assume that $\sum_{x=0}^{\infty} \eta_x \rho^x < \infty$. Since we will be dealing with a continuum of choices for $\rho$, it will be useful to designate all of the frog distributions that will guarantee transience in the frog model with drift. Hence, we define the following set of integer sequences:

$$\mathbb{H} := \left\{ \eta \in \mathbb{N}^{\mathbb{N} \cup \{0\}} : \sum_{x=0}^{\infty} \eta_x \rho^x < \infty \text{ for all } \rho \in (0,1) \right\}.$$

It's worth noting that $\mathbb{H}$ does contain unbounded elements, such as $\eta = \{1, 2, 3, 4, \ldots\}$. In fact, any integer sequence $\eta$ such that $\eta_x = o(\alpha^x)$ as $x \to \infty$ for any $\alpha > 1$ is in $\mathbb{H}$.

Similarly to the single-frog case, we can construct a family of independent random variables $(X_{\rho,\eta})_{\rho \in (0,1), \eta \subset \mathbb{H}}$ on the probability space $(\Omega, \mathcal{F}, P)$ such that for each $\rho \in (0,1)$ and $\eta \in \mathbb{H}$, $X_{\rho,\eta}$ shares the same distribution as the negative of the minimum of the frog model with drift $\rho$ and frog distribution $\eta$.

By using similar ideas as in the single-frog case, we can find the distribution of $X_{\rho,\eta}$:

$$P(X_{\rho,\eta} \leq x) = \prod_{k=0}^{\infty} \left(1 - \rho^{x+k+1}\right)^{\eta_k}. \tag{13}$$

For simplicity, we will define the integer sequence $\{\Delta_k\}_{k=0}^{\infty}$ by $\Delta_0 = \eta_0$ and for all $k \geq 1$, $\Delta_k = \eta_k - \eta_{k-1}$. The density of $X_{\rho,\eta}$ is then

$$
\begin{aligned}
P(X_{\rho,\eta} = x) &= \prod_{k=0}^{\infty} \left(1 - \rho^{x+k+1}\right)^{\eta_k} - \prod_{k=0}^{\infty} \left(1 - \rho^{x+k}\right)^{\eta_k} \\
&= \prod_{k=0}^{\infty} \left(1 - \rho^{x+k+1}\right)^{\eta_k} \left(1 - \prod_{k=0}^{\infty} (1 - \rho^{x+k})^{\Delta_k}\right).
\end{aligned} \tag{14}
$$

One special case to consider is $\eta_x = n \in \mathbb{N}$ for all $x \geq 0$. Then, (13) and (14) become

$$P(X_{\rho,\eta} \leq x) = \prod_{k=0}^{\infty} (1 - \rho^{x+k+1})^n = (\rho^{x+1}; \rho)_{\infty}^n,$$

$$P(X_{\rho,\eta} = x) = \left(1 - (1 - \rho^x)^n\right)(\rho^{x+1}; \rho)_{\infty}^n.$$

With frog distributions $\eta$ that differ from the single-frog case, we could assume that the moments of $X_{\rho,\eta}$ grow at different rates than $\frac{\ln(1-\rho)}{\ln \rho}$ found in Corollary 3.5. However, by the theorem below, if $\eta$ grows at a "slow enough" rate, the moments of $X_{\rho,\eta}$ will behave asymptotically similar to those of the single-frog case.

8

**Theorem 4.1.** *For each $\rho \in (0,1)$, let $z_\rho = \frac{\ln(1-\rho)}{\ln \rho}$. Suppose that $\{\eta_k\}_{k=0}^\infty \in \mathbb{H}$ is such that $\lim_{\rho \uparrow 1}(1-\rho)^{1+\delta} \sum_{k=0}^\infty \eta_k \rho^k = 0$ for all $\delta > 0$. Then, for all $m \in \mathbb{N}$, $E(X_{\rho,\eta}^m) \sim z_\rho^m$ as $\rho \uparrow 1$.*

The proof of the theorem is given below in this section, after a short discussion of the result. Note that according to the theorem, any even frog distribution $\eta = \{n, n, \dots\}$, where $n \in \mathbb{N}$, will produce the same asymptotic rate for the moments. Not only that, but there exist unbounded choices for $\eta$ that produce the same rate as well. One simple example of such a choice is $\eta = \{1, 2, 3, \dots\}$.

A consequence of Theorem 4.1 is an analogue to Theorem 3.6 in the single-frog case revealing that $\frac{\ln(1-\rho)}{\ln \rho}$ is also an appropriate scaling for $X_{\rho,\eta}$'s convergence in probability as $\rho \uparrow 1$.

**Theorem 4.2.** *Let $\eta \in \mathbb{H}$ satisfy the conditions of Theorem 4.1. Let $Y_{\rho,\eta}$ share the same distribution as $\frac{\ln \rho}{\ln(1-\rho)} X_{\rho,\eta}$ for each $\rho \in (0,1)$. Then, as $\rho \uparrow 1$, $Y_{\rho,\eta} \to 1$ in probability.*

To clarify the intuition behind this result it is instructive to consider the case of an even frog configuration $\eta_x = m$ for all $x \geq 0$, where $m \in \mathbb{N}$ is a fixed integer, and observe that the corresponding model can be thought of as a composition of $m$ independent models with $\eta_x = 1$. In this case, Theorem 4.2 is a direct implication of the result in Theorem 3.6 following by a simple observation that since the random variable $Y_\rho$ is asymptotic to a constant, the same is true for its analogue $Y_{\rho,\eta}$ in Theorem 4.2 which is the minimum of $m$ independent copies of $Y_\rho$. From this perspective, Theorems 4.1 and Theorem 4.2 can be viewed as an indirect extension of this argument to sequences $\eta_x$ growing sufficiently slowly, so that they can be well enough approximated by initial configurations with an even distribution of frogs (notice that the further is the initial placement of a frog from the origin the less relevant it is for the asymptotic of $Y_{\rho,\eta}$).

**Remark 4.3.** *The proof of Proposition 3.8 can be carried over, and hence its conclusion remains valid, for sequences $\eta \in \mathbb{H}$ that satisfy the conditions of Theorem 4.1.*

Before we begin the proof of Theorem 4.1, we must first introduce a lemma.

**Lemma 4.4.** *For each $\rho \in (0,1)$, let $z_\rho = \frac{\ln(1-\rho)}{\ln \rho}$. Then, for all $\delta > 0$ and $m \in \mathbb{N}$,*

$$\sum_{x=1}^\infty \big(z_\rho(1+\delta) + x\big)^m \rho^x \sim \frac{z_\rho^m(1+\delta)^m}{1-\rho} \qquad as \qquad \rho \uparrow 1. \tag{15}$$

*Proof of Lemma 4.4.* For the quotient of the left- and right-hand sides in (15) we have

$$(1-\rho)\sum_{x=1}^\infty \Big(1 + \frac{x}{z_\rho(1+\delta)}\Big)^m \rho^x = (1-\rho)\sum_{x=1}^\infty \rho^x \sum_{j=0}^m \binom{m}{j} \frac{x^j}{z_\rho^j(1+\delta)^j}$$

$$= (1-\rho)\sum_{j=0}^m \frac{1}{z_\rho^j(1+\delta)^j} \binom{m}{j} \Big(\sum_{x=1}^\infty x^j \rho^x\Big) \leq (1-\rho)\sum_{j=0}^m \frac{1}{z_\rho^j(1+\delta)^j} \binom{m}{j} \frac{j!}{(1-\rho)^{j+1}}$$

$$= \sum_{j=0}^m \frac{m!}{(m-j)!} \frac{1}{[z_\rho(1+\delta)(1-\rho)]^j}.$$

Now, as $\rho \uparrow 1$, $\frac{\ln(\rho)}{1-\rho} \to -1$ and $\ln(1-\rho) \to -\infty$. Thus, $z_\rho(1-\rho) \to \infty$. Hence, for any $j > 0$, the $j^{\text{th}}$ term of the above sum goes to 0 as $\rho \uparrow 1$. So when taking the limit, the only term in the sum that survives would be the $0^{\text{th}}$, which is equal to 1 for all $\rho$. $\square$

We now proceed with the proof of the above theorem.

*Proof of Theorem 4.1.* Since the frog model corresponding to $X_{\rho,\eta}$ contains more frogs than the single-frog case, by Corollary 3.5, $\liminf_{\rho\uparrow 1} \frac{E(X_{\rho,\eta}^m)}{z_\rho^m} \geq 1$. For the other inequality, note that for any $\eta \in \mathbb{H}$, $E(X_{\rho,\eta}^m) \leq E(X_{\rho,\theta}^m)$, where $\theta_k = \max\{\eta_j : j = 1, 2, \ldots, k\}$. Thus, we can assume without loss of generality that $\eta$ is a nondecreasing sequence, and hence $\Delta_k \geq 0$ for all $k$.

Choosing $\delta > 0$, we consider the sum for $E(X_{\rho,\eta}^m)$ and split it at the point $\lfloor z_\rho(1+\delta) \rfloor$. For the tail sum, we find that

$$\sum_{x=\lfloor z_\rho(1+\delta)\rfloor+1}^{\infty} x^m P(X_{\rho,\eta}=x) \leq \sum_{x=\lfloor z_\rho(1+\delta)+1\rfloor}^{\infty} x^m\left(1-\prod_{k=0}^{\infty}(1-\rho^{x+k})^{\Delta_k}\right)$$

$$\sim \sum_{x=1}^{\infty}\left(z_\rho(1+\delta)+x\right)^m\left(1-\prod_{k=0}^{\infty}\left(1-\rho^{z_\rho(1+\delta)}\rho^{x+k}\right)^{\Delta_k}\right).$$

Since $\rho^{z_\rho} = (1-\rho)$, expanding the infinite product up to the first-order terms, we obtain:

$$\sum_{x=\lfloor z_\rho(1+\delta)\rfloor+1}^{\infty} x^m P(X_{\rho,\eta}=x) \leq \sum_{x=1}^{\infty}\left(z_\rho(1+\delta)+x\right)^m\left(1-\prod_{k=0}^{\infty}\left(1-(1-\rho)^{1+\delta}\rho^{x+k}\right)^{\Delta_k}\right)$$

$$\leq \sum_{x=1}^{\infty}\left(z_\rho(1+\delta)+x\right)^m\left(\sum_{k=0}^{\infty}\Delta_k(1-\rho)^{1+\delta}\rho^{x+k}\right)$$

$$= (1-\rho)^{1+\delta}\left(\sum_{k=0}^{\infty}\Delta_k\rho^k\right)\sum_{x=1}^{\infty}\left(z_\rho(1+\delta)+x\right)^m\rho^x \sim (1-\rho)^{1+\delta}\left(\sum_{k=0}^{\infty}\eta_k\rho^k\right)z_\rho^m(1+\delta)^m.$$

The equivalence result in last line comes from Lemma 4.4, combined with the fact that $\sum_{k=0}^{\infty}\Delta_k\rho^k = \eta_0 + (1-\rho)\sum_{k=1}^{\infty}\eta_k\rho^k$. We also derive the following upper bound for the finite sum:

$$\sum_{x=0}^{\lfloor z_\rho(1+\delta)\rfloor} x^m\, P(X_{\rho,\eta}=x) \leq \lfloor z_\rho\rfloor^m(1+\delta)^m P\left(X_{\rho,\eta}\leq z_\rho(1+\delta)\right) \leq z_\rho^m(1+\delta)^m.$$

Thus, we can derive an upper bound for the following limit:

$$\limsup_{\rho\uparrow 1} \frac{E(X_{\rho,\eta}^m)}{z_\rho^m} \leq \lim_{\rho\uparrow 1}(1+\delta)^m\left(1+(1-\rho)^{1+\delta}\cdot\sum_{k=0}^{\infty}\eta_k\rho^k\right) = (1+\delta)^m.$$

Since $\delta > 0$ is arbitrary, $\limsup_{\rho\uparrow 1}\frac{E(X_{\rho,\eta}^m)}{z_\rho^m} \leq 1$, and this completes the proof. $\qquad\square$

# 5 Initial configuration $\eta$ supported on the whole $\mathbb{Z}$

In this section we will provide asymptotic bounds for the minimum of the frog model's range under the assumption that $\eta$ is supported on all of $\mathbb{Z}$. The main result of this section is stated in Theorem 5.1 below.

Up until now, we have assumed that there were no sleeping frogs on any of the negative sites. With this assumption, all of the frogs on $\mathbb{Z}_+$ would eventually wake w.p.1, and we only needed to observe the collective minima of those frogs. However, when we consider the transient frog model with configuration $\eta$ supported on all of $\mathbb{Z}$, we now have a random number of active frogs originating from the negative sites that have the potential of expanding the range. We begin to explore the moments of the minimum of this case by the groundwork laid in the previous sections for the $\eta$ supported only on nonnegative sites.

Fix any $n \in \mathbb{N}$. We will assume throughout this section that $\eta_x = n$ for any $x \in \mathbb{Z}$, that is exactly $n$ frogs are initially placed at each site of $\mathbb{Z}$. Our proofs in this section rely on the following description of the "avalanche structure" of the model. We refer to the frogs initially located in the nonnegative sites of $\mathbb{Z}$ as the "first wave". If we just observe the nonnegative frogs, we can locate the leftmost site visited by the frogs from the first wave. We consider the frogs on the negative sites down to the minimum ever visited by the first wave to be the "second wave" of frogs being activated. Tracking the leftmost site visited by the frogs from the second wave, we designate a "third wave" of frogs activated between the subsequent minimums. We will continue to label these activated frogs in terms of waves. Since we assume a transient model, there will eventually be a final wave of frogs w.p.1 that never venture any more to the left than their initial locations. In this section, let the negative of the leftmost site visited by any of the active frogs be $\widetilde{X}_{\rho,n}$.

The following theorem provides upper and lower bounds for the $m^{\text{th}}$ moment of $\widetilde{X}_{\rho,n}$ for any given $m \in \mathbb{N}$. While the bounds contain the familiar $z_\rho = \frac{\ln(1-\rho)}{\ln(\rho)}$ term from Sections 3 and 4, they are not immediately obvious from the previous results.

**Theorem 5.1.** *The following holds for any $m \in \mathbb{N}$:*

(a) *The $m^{th}$ moment of $\widetilde{X}_{\rho,n}$ is bounded above by a function $\phi : (0,1) \to [0,\infty)$ such that*

$$\phi(\rho) \sim z_\rho^m \left(\frac{1-\rho}{2\pi}\right)^{\frac{n}{2}} \exp\left\{\frac{\pi^2}{6} \frac{mn}{1-\rho}\right\} \quad as \quad \rho \uparrow 1. \qquad (16)$$

(b) *For any function $\delta : (0,1) \to (0,1)$ such that $\lim_{\rho \uparrow 1} \delta(\rho) = 0$ and $\lim_{\rho \uparrow 1} (1-\rho)^{\delta(\rho)} = 0$, the $m^{th}$ moment of $\widetilde{X}_{\rho,n}$ is bounded below by a function $\psi_\delta : (0,1) \to [0,\infty)$ such that*

$$\psi_\delta(\rho) \sim m! \, z_\rho^m \exp\left\{\frac{mn}{(1-\rho)^{\delta(\rho)}}\right\} \quad as \quad \rho \uparrow 1. \qquad (17)$$

**Remark 5.2.** *An example of a function $\delta$ that satisfies the conditions of Theorem 5.1 (b) is $\delta(\rho) = \left\{\ln\left(\frac{1}{1-\rho}\right)\right\}^{-\alpha}$ for a constant $\alpha \in (0,1)$ and $\rho > 1 - e^{-1}$ (as long as we are only interested in the asymptotic as $\rho \uparrow 1$, the values of $\delta(\rho)$ can be assigned arbitrarily for $\rho \leq 1 - e^{-1}$). In this case, $\psi_\delta(\rho) \sim m! \, z_\rho^m \exp\left\{mn \exp\left(|\ln(1-\rho)|^{1-\alpha}\right)\right\}$ as $\rho \uparrow 1$. For the sake of comparison with the upper bound, notice that the latter can be written as $\phi(\rho) \sim z_\rho^m \left(\frac{1-\rho}{2\pi}\right)^{\frac{n}{2}} \exp\left\{mn \exp\left(|\ln(1-\rho)|\right)\right\}$ as $\rho \uparrow 1$, and that the parameter $\alpha \in (0,1)$ can be chosen arbitrarily close to zero.*

To motivate and clarify the intuition behind the coupling construction employed in the proof of Theorem 5.1 given below, we precede the proof by the following observation.

**Remark 5.3.** *Consider the model described in Section 3, namely $\eta_x = 1$ for $x \geq 0$ and $\eta_x = 0$ for $x < 0$. Fix any $\delta > 0$ and consider the probability $P_{\rho,\delta}$ that no one of the frogs initially placed to the right of $z_\rho(1+\delta)$ will ever reach zero. For simplicity and without loss of generality we will treat $z_\rho(1+\delta)$ as an integer. Then, since $\rho^{z_\rho(1+\delta)} = (1-\rho)^{1+\delta}$, we have $P_{\rho,\delta} = \prod_{j=1}^{\infty}\bigl(1 - \rho^{z_\rho(1+\delta)+j}\bigr) = \prod_{j=1}^{\infty}\bigl(1 - (1-\rho)^{1+\delta}\rho^j\bigr)$. Since $1 - x > e^{-2x}$ for all $x > 0$ small enough, we obtain that $P_{\rho,\delta} \geq \exp\bigl\{-2(1-\rho)^{1+\delta}\sum_{j=1}^{\infty}\rho^j\bigr\} = \exp\bigl\{-2\rho(1-\rho)^\delta\bigr\}$ for all values of $\rho$ sufficiently close to 1, and hence $\lim_{\rho\uparrow 1} P_{\rho,\delta} = 1$.*

*Now, recall that the results of Section 3 indicate tight concentration of the distribution of $X_\rho$ around $z_\rho$ as $\rho \uparrow 1$. On the other hand, heuristically, $\lim_{\rho\uparrow 1} Q_{\rho,\delta} = 0$ indicates that for large values of $\rho$ only the first $z_\rho$ frogs are relevant to the dynamics of the model. To further support this claim, consider the probability $Q_{\rho,\delta}$ that no one of the $z_\rho(1-\delta)$ frogs initially placed at the first $z_\rho(1-\delta)$ nonnegative integers will ever reach $-z_\rho(1-\delta)$. Similarly as before, we treat $z_\rho(1-\delta)$ as an integer. Then*

$$
\begin{aligned}
Q_{\rho,\delta} &= \prod_{j=0}^{z_\rho(1-\delta)-1}\bigl(1 - \rho^{z_\rho(1-\delta)+j}\bigr) \leq \exp\Bigl\{-(1-\rho)^{1-\delta}\sum_{j=0}^{z_\rho(1-\delta)-1}\rho^j\Bigr\} \\
&= \exp\Bigl\{-(1-\rho)^{-\delta}\bigl(1 - \rho^{z_\rho(1-\delta)}\bigr)\Bigr\} = \exp\Bigl\{-(1-\rho)^{-\delta}\bigl(1 - (1-\rho)^{1-\delta}\bigr)\Bigr\} \\
&\leq \exp\Bigl\{-(1-\rho)^{-\delta}\bigl(1 - e^{-\rho(1-\delta)}\bigr)\Bigr\}.
\end{aligned}
$$

*Thus $\lim_{\rho\uparrow 1} Q_{\rho,\delta} = 0$.*

*Heuristically, $\lim_{\rho\uparrow 1} P_{\rho,\delta} = 1$ along with $\lim_{\rho\uparrow 1} Q_{\rho,\delta} = 0$ tell us that the dynamics of the model considered in Section 3 is for large values of $\rho$ similar to the dynamics of a modification where $\eta_x$ equals 1 only if $0 \leq x < z_\rho$ and is 0 otherwise. The proof of Theorem 5.1 given below is using an interpretation of the model considered in this section as an "avalanche" of the models described in Section 4 and is using the above heuristic observation to produce upper and lower bounds of Theorem 5.1 for the model range.*

*Proof of Theorem 5.1.*
(a) We first provide an upper bound for the moments of $\widetilde{X}_{\rho,n}$ by coupling the frog model with the following variant. First observe the minimum location reached by the frogs that are initially placed on the nonnegative sites and obtain a second wave of active frogs. Modify the original second wave in the following way. Put $n$ more frogs on each site to the right of the minimum reached by the first wave and suppose that only the second wave can activate them. Note that without the consideration of activation times, this set of frogs is a translation of the configuration of nonnegative frogs considered in Sections 3 and 4. Find the minimum bound for this modified configuration and, for the resulting next wave, add frogs to all of the right-side sites in a similar fashion. Since there is a positive probability for the nonnegative frogs never reach $-1$, and each wave of frogs is a translation of this case, the waves will terminate w.p.1. Let $W_{\rho,n}$ be the negative of the minimum bound for this model.

A formal definition of the distribution of $W_{\rho,n}$ can be given in the following manner. Define the sequence $\widetilde{\eta} = \{\widetilde{\eta}_x\}_{x\in\mathbb{Z}} \in \mathbb{Z}_+^{\mathbb{Z}}$ as follows: $\eta_x = n$ if $x \geq 0$ and $\eta_x = 0$ if $x < 0$. Such a configuration has been considered in Section 4. Let $X_{\rho,n}$ be the negative of the range of the corresponding model and let $X_{\rho,n}^{(k)}$, $k \in \mathbb{Z}$, be independent copies of this random variable.

Let $T_{\rho,n} = \inf\{k \in \mathbb{N} : X_{\rho,n}^{(k)} = 0\}$ and $W_{\rho,n} = \sum_{k=1}^{T_{\rho,n}-1} X_{\rho,n}^{(k)}$, where, as usual, the empty sum (when $T_{\rho,n} = 1$) is interpreted as zero.

To facilitate our computations we will actually use the following equivalent modification of this definition. Let $\varepsilon_{\rho,n} = P(X_{\rho,n} = 0) = \big((\rho;\rho)_\infty\big)^n$, where $X_{\rho,n}$ is the minimum of the range of the case with $n$ frogs on each nonnegative site, introduced in Section 4. Thus $\varepsilon_\rho = P\big(X_{\rho,n}^{(k)} = 0\big)$ for any $k \in \mathbb{N}$. Let $\widetilde{T}_{\rho,n}$ be a geometric random variable with parameter $\varepsilon_\rho$. Namely, $P\big(\widetilde{T}_{\rho,n} = k\big) = (1-\varepsilon_\rho)^k \varepsilon_\rho$, $k = 0, 1, \ldots$. Notice that, $\lim_{\rho \to 1} \varepsilon_{\rho,n} = 0$, and hence $\widetilde{T}_{\rho,n}$ converges to infinity in probability as $\rho \uparrow 1$. Clearly, $\widetilde{T}_{\rho,n}$ has the same distribution as $T_{\rho,n} - 1$. Let $\mathcal{Y}_{\rho,n} = \{Y_{\rho,n}^{(k)}\}_{k \in \mathbb{Z}}$ be an i.i.d. sequence independent of $\widetilde{T}_{\rho,n}$ and such that for any $j \in \mathbb{N}$, $P(Y_{\rho,n} = j) = P(X_{\rho,n} = j | X_{\rho,n} > 0) = \frac{P(X_{\rho,n}=j)}{1-P(X_{\rho,n}=0)}$. Finally, let $\widetilde{W}_{\rho,n} = \sum_{k=1}^{\widetilde{T}_{\rho,n}} Y_{\rho,n}^{(k)}$. Clearly, $\widetilde{W}_{\rho,n}$ has the same distribution as $W_{\rho,n}$.

For $m \in \mathbb{N}$, we look at the $m^{\text{th}}$ moment of $\widetilde{W}_{\rho,n}$ conditioned on $\widetilde{T}_{\rho,n}$:

$$
E\big(\widetilde{W}_{\rho,n}^m | \widetilde{T}_{\rho,n}\big) = E\bigg[\bigg(\sum_{k=1}^{\widetilde{T}_{\rho,n}} Y_{\rho,n}^{(k)}\bigg)^m \bigg| \widetilde{T}_{\rho,n}\bigg] \le \bigg(\sum_{k=1}^{\widetilde{T}_{\rho,n}} E\big[(Y_{\rho,n}^{(k)})^m\big]^{\frac{1}{m}}\bigg)^m = \big(\widetilde{T}_{\rho,n}\big)^m E\big[(Y_{\rho,n}^{(1)})^m\big],
$$

where we use Minkowski inequality and the fact that $Y_{\rho,n}^{(k)}$ are independent of $\widetilde{T}_{\rho,n}$. From this conditioned expectation, we approximate the $m^{\text{th}}$ moments of $W_{\rho,n}$ as $\rho \uparrow 1$:

$$
E\big(W_{\rho,n}^m\big) = E\big(E\big(\widetilde{W}_{\rho,n}^m | \widetilde{T}_{\rho,n}\big)\big) \le E\big[\big(\widetilde{T}_{\rho,n}\big)^m\big] \cdot E\big[\big(Y_{\rho,n}^{(1)}\big)^m\big] \sim \varepsilon_{\rho,n}^{-m} z_\rho^m, \tag{18}
$$

where $z_\rho = \frac{\ln(1-\rho)}{\ln \rho}$ as in Section 4. To evaluate the moments of $\widetilde{T}_{\rho,n}$ we used the following known result whose short proof is supplied for reader's convenience.

**Lemma 5.4.** *For $\varepsilon \in (0,1)$, let $T_\varepsilon$ be a geometric random variable with parameter $\varepsilon_\rho$. Namely, $P\big(T_\varepsilon = k\big) = (1-\varepsilon)^k \varepsilon$, $k = 0, 1, \ldots$. Then, for any $m \in \mathbb{N}$, $E(T_\varepsilon^m) \sim \varepsilon^{-m}$ as $\varepsilon \uparrow 1$.*

*Proof of Lemma 5.4.* We have as $\varepsilon \uparrow 1$,

$$
E(T_\varepsilon^m) = \int_0^\infty P(T_\varepsilon^m > x)\, dx = \int_0^\infty P(T_\varepsilon > y) \cdot m y^{m-1}\, dy \sim \int_0^\infty (1-\varepsilon)^{y+1} \cdot m y^{m-1}\, dy
$$

$$
= m(1-\varepsilon) \int_0^\infty e^{\ln(1-\varepsilon)y} y^{m-1}\, dy = m(1-\varepsilon) \frac{\Gamma(m)}{|\ln(1-\varepsilon)|^m} \sim \frac{m!}{\varepsilon^m}.
$$

The proof of the lemma is complete. $\qquad\square$

Using (18), we finally arrive at the asymptotic bound in *(a)* through the asymptotic of the $q$-Pochhammer symbol derived from [14]: for $\rho = e^{-t}$, as $t \downarrow 0$, $(\rho;\rho)_\infty \sim \sqrt{\frac{2\pi}{t}} \exp\big(\frac{-\pi^2}{6t}\big)$. Note that $t = -\ln \rho \sim 1 - \rho$, and we have *(a)*. $\qquad\square$

*(b)* Now, consider another variant of the frog model. Define a function $\delta(\rho) : (0,1) \to (0,1)$. To simplify notation, we will occasionally use $\delta_\rho$ for $\delta(\rho)$. For a given $\rho$, consider the configuration $\hat{\eta}$ with $\hat{\eta}_k = n$ if $k \in \{0, 1, 2, \ldots, z_\rho(1-\delta_\rho)\}$ and $\hat{\eta}_k = 0$ elsewhere. For simplicity and without loss of generality, we will assume that $z_\rho(1-\delta_\rho)$ is integer-valued. Start the model, and see if the frogs eventually reach the site $-z_\rho(1-\delta_\rho)$. If they do, activate $n$ frogs

on each of the $z_\rho(1 - \delta_\rho)$ sites to the left of the origin. Now observe if the newly activated frogs reach the site $-2z_\rho(1 - \delta_\rho)$. If they do, activate $n$ frogs on all the $z_\rho(1 - \delta_\rho)$ sites to the left of the sites previously activated. Continue this procedure indefinitely. Note that at any observed time step, this model will always have fewer active frogs than the frog model with initial configuration $\eta$ with $\eta_k = n$ for all $k \in \mathbb{Z}$. By Theorem 2.1 in [12], the frog model with this configuration is transient. Hence, the variant model is transient, and the process will eventually stop producing new sets of frogs from the left. Let $V_{\rho,\delta}$ be the negative of the minimum of the range of the variant, and let $\tau_{\rho,\delta}$ be the number of activated blocks of the length $z_\rho(1 - \delta_\rho)$, not including the initial one located within $\mathbb{Z}_+$.

Let $\theta_{\rho,\delta} = \prod_{j=0}^{z_\rho(1-\delta_\rho)-1}\left(1 - \rho^{z_\rho(1-\delta_\rho)}\rho^j\right)^n = \left((\rho^{z_\rho(1-\delta_\rho)};\rho)_{z_\rho(1-\delta_\rho)}\right)^n$. Thus $\theta_{\rho,\delta}$ is the probability that none of the frogs located at the first $z_\rho(1 - \delta_\rho)$ nonnegative sites will ever reach the half-line on the left of $-z_\rho(1 - \delta_\rho)$. Viewing the event of a block of active frogs reaching $z_\rho(1 - \delta_\rho)$ units to the left as a "failure", we see that $\tau_{\rho,\delta}$ is geometrically distributed with parameter $\theta_{\rho,\delta}$. Namely, $P(\tau_{\rho,\delta} = k) = (1 - \theta_{\rho,\delta})^k \theta_{\rho,\delta}$, $k = 0, 1, \ldots$. Since $\rho^{z_\rho} = 1 - \rho$,

$$
\begin{aligned}
\theta_{\rho,\delta} &= \prod_{j=0}^{z_\rho(1-\delta_\rho)-1}\left(1 - (1-\rho)^{1-\delta_\rho}\rho^j\right)^n \leq \exp\left(-n(1-\rho)^{1-\delta_\rho}\sum_{j=0}^{z_\rho(1-\delta_\rho)-1}\rho^j\right) \\
&= \exp\left(-n\frac{1 - \rho^{z_\rho(1-\delta_\rho)}}{(1-\rho)^{\delta_\rho}}\right) = \exp\left(-n\frac{1 - (1-\rho)^{1-\delta_\rho}}{(1-\rho)^{\delta_\rho}}\right).
\end{aligned}
$$

With the constraints specified in the theorem's hypotheses, $\theta_{\rho,\delta} \to 0$ as $\rho \uparrow 1$. Hence, by Lemma 5.4, $E(\tau_{\rho,\delta}^m) \sim m!\,\theta_{\rho,\delta}^{-m}$ as $\rho \uparrow 1$. Clearly, $\widetilde{X}_{\rho,n}$ is stochastically dominated from below by $V_{\rho,\delta} = \tau_{\rho,\delta} \cdot z_\rho(1 - \delta_\rho)$. The lower bound for the moments of $\widetilde{X}_{\rho,n}$ is therefore

$$
E(\tau_{\rho,\delta}^m) \cdot z_\rho^m(1 - \delta_\rho)^m \sim m!\,z_\rho^m\exp\left\{mn\frac{1 - (1-\rho)^{1-\delta_\rho}}{(1-\rho)^{\delta_\rho}}\right\} \sim m!\,z_\rho^m\exp\left\{\frac{mn}{(1-\rho)^{\delta_\rho}}\right\},
$$

as $\rho \uparrow 1$. The proof of the theorem is complete. $\qquad\square$

# References

[1] O. S. M. Alves, F. P. Machado, K. Ravishankar, and S. Yu. Popov, *The shape theorem for the frog model with random initial configuration*, Markov Process. Related Fields **7** (2001), 525–539.

[2] O. S. M. Alves, F. P. Machado, and S. Yu. Popov, *The shape theorem for the frog model*, Ann. Appl. Probab. **12** (2002), 533–546.

[3] O. S. M. Alves, F. P. Machado, and S. Yu. Popov, *Phase transition for the frog model*, Electron. J. Probab. **7** (2002), 1–21.

[4] G. E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra (CBMS Regional Conference Series in Mathematics, Vol. 66)*, AMS, Providence, RI, 1986.

[5] D. Bertacchi, F. P. Machado, and F. Zucca, *Local and global survival for nonhomogeneous random walk systems on* $\mathbb{Z}$, Adv. in Appl. Probab. **46** (2014), 256–278.

[6] C. A. Charalambides, *Enumerative Combinatorics*, Discrete Mathematics and Its Applications Series, Chapman and Hall/CRC, 2002.

[7] F. Comets, J. Quastel, and A. F. Ramirez, *Fluctuations of the front in a stochastic combustion model*, Ann. Inst. Henri Poincaré Probab. Stat. **43** (2007), 147–162.

[8] C. Döbler and L. Pfeifroth, *Recurrence for the frog model with drift on* $\mathbb{Z}^d$, Electron. Commun. Probab. **19** (2014), 1-13.

[9] R. Durrett, *Probability: Theory and Examples*, 4th ed., Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2010.

[10] T. Ernst, *A Comprehensive Treatment of q-Calculus*, Birkhäuser/Springer, Basel, 2012.

[11] P. Eschenfeldt, B. Gross, and N. Pippenger, *Asymptotic behavior of the moments of the maximum queue length during a busy period*, available at arXiv:1106.3590, 2011.

[12] N. Gantert and P. Schmidt, *Recurrence for the frog model with drift on* $\mathbb{Z}$, Markov Process. Related Fields **15** (2009), 51–58.

[13] G. Gaspar and M. Rahman, *Basic Hypergeometric Series (Encyclopedia of Mathematics and its Applications, Book 96)*, 2nd ed., Cambridge University Press, 2004.

[14] B. Gordon and R. J. McIntosh, *Some eighth order mock theta functions*, J. London Math. Soc. **62** (2000), 321–335.

[15] C. Hoffman, T. Johnson, and M. Junge, *Recurrence and transience for the frog model on trees*, available at arXiv:1404.6238, 2014. To apper in the Ann. Probab.

[16] C. Hoffman, T. Johnson, and M. Junge, *From transience to recurrence with Poisson tree frogs*, available at arXiv:1501.05874, 2014. To appear in the Ann. Appl. Probab.

[17] W. P. Johnson, *The curious history of Faà di Bruno's formula*, Amer. Math. Monthly **109** (2002), 217–234.

[18] E. Lebensztayn, F. P. Machado, and M. Z. Martinez, *Random walks systems with killing on* $\mathbb{Z}$, Stochastics **80** (2008), 451–457.

[19] E. Lebensztayn, F. P. Machado, and M. Z. Martinez, *Nonhomogeneous random walks systems on* $\mathbb{Z}$, J. Appl. Probab. **47** (2010), 562–571.

[20] E. Lukacs *Applications of Faà di Bruno's formula in mathematical statistics*, Amer. Math. Monthly **62** (1955), 340–348.

[21] G. Peccati and M. Taqqu, *Wiener Chaos: Moments, Cumulants and Diagrams. A Survey with Computer Implementation* (Bocconi&Springer Series, Book 1), Springer, 2011.

[22] S. Yu. Popov, *Frogs in random environment*, J. Stat. Physics **102** (2001), 191–201.

[23] S. Yu. Popov, *Frogs and some other interacting random walks models*, Discrete Math. Theor. Comput. Sci. Proc., AC, 2003, 277–288.

[24] A. F. Ramirez and V. Sidoravicius, *Asymptotic behavior of a stochastic combustion growth process*, J. Eur. Math. Soc. **6** (2004), 293–334.

[25] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, 1976.

[26] A. Telcs and N. Wormald, *Branching and tree-indexed random walks on fractals*, J. Appl. Probab. **36** 1999, 999–1011