On Wallis-type products and Pólya's urn schemes

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Abstract

A famous "curious identity" of Wallis gives a representation of the constant π in terms of a simply structured infinite product of fractions. Sondow and Yi [Amer. Math. Monthly **117** (2010), 912-917] identified a general scheme for evaluating Wallis-type infinite products. The main purpose of this paper is to discuss an interpretation of the scheme by means of Pólya urn models.

1 Introduction

This paper is motivated by the work of Sondow and Yi [20], where several examples of Wallis-type products (see Definition 2.6 below) have been constructed. The method used in [20] yields "cyclically structured" converging infinite products of fractions and evaluates their limit by means of the gamma function (see Section 2 below). Only in a limited range of cases is an expression of the limit in terms of powers of π and algebraic numbers known. Section 3 contains an instructive survey of generic examples. In Section 4 we discuss a relation between the Wallis-type products and the Pólya urn scheme. Our main observation is contained in the statement of Theorem 4.4. Throughout the paper, $\Gamma(\,\cdot\,)$ is the gamma function and $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are, respectively, natural, rational, real, and complex numbers.

2 Wallis-type infinite products

Our starting point is a representation of the constant π discovered by Wallis [3, p. 68]:

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\frac{10}{9}\frac{10}{11}\frac{12}{11}\frac{12}{13}\cdots. \tag{1}$$

A standard proof of the identity (1) relies on the evaluation of Wallis's integral $\int_0^{\pi/2}\sin^{2n+1}t\,dt$ [2]. A relation of Wallis's product to Euler's and Leibniz's formulas $\frac{\pi^2}{6}=\sum_{n=1}^\infty\frac{1}{n^2}$ and $\frac{\pi}{4}=\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}$ is discussed, for instance, in [17, 25]. The cyclic structure of (1) is formalized as follows:

$$\frac{\pi}{2} = \prod_{n=0}^{\infty}\frac{2n+2}{2n+1}\frac{2n+2}{2n+3}. \tag{2}$$

This is generalized in the next proposition, which is a slight variation of a part of [20, Theorem 1]. The result is the following Euler's formula [8, Section VII.6] applied to fractions:

$$\frac{\sin(\pi x)}{\pi x} = \prod_{j=1}^{\infty}\Big(1-\frac{x^2}{j^2}\Big) = \prod_{n=0}^{\infty}\Big(1-\frac{x^2}{(n+1)^2}\Big), \qquad \text{for } x\in\mathbb{R}. \tag{3}$$

Proposition 2.1. *For any $k,m\in\mathbb{N}$ such that $m<k$, it follows that*

$$\frac{\pi m/k}{\sin(\pi m/k)} = \prod_{n=0}^{\infty}\frac{nk+k}{nk+k-m}\,\frac{nk+k}{nk+k+m}. \tag{4}$$

Example 2.2.

(a) Letting $k=6$ and $m=1$ in (4), we obtain $\prod_{n=0}^{\infty}\frac{6n+6}{6n+5}\frac{6n+6}{6n+7}=\frac{\pi}{3}$ [25, p. 187].

(b) The identity $\cos\pi x-\cos\pi y = 2\sin\frac{\pi(y-x)}{2}\sin\frac{\pi(y+x)}{2}$ with $x=\frac{1}{6}, y=\frac{1}{4}$ and (4) yield

$$\begin{aligned}\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}} &= 2\sin\frac{\pi}{24}\sin\frac{5\pi}{24}\\ &= \frac{10\pi^2}{24^2}\prod_{n=0}^{\infty}\frac{24n+19}{24n+24}\,\frac{24n+23}{24n+24}\,\frac{24n+25}{24n+24}\,\frac{24n+29}{24n+24}.\end{aligned}$$

We remark that (3) can be thought of as the representation of the infinite Taylor polynomial

$$\frac{\sin(\pi x)}{\pi x} = 1-\frac{(\pi x)^2}{3!}+\frac{(\pi x)^4}{5!}-\frac{(\pi x)^6}{7!}+\dots$$

as $\prod_{j=1}^{\infty}\big(1-\frac{x}{x_j}\big)\big(1-\frac{x}{x_{-j}}\big)$, where $x_j=j$ are the roots of the polynomial [17, Chapter II].

The next proposition is a consequence of the following counterpart of (3) for the cosine function:

$$\cos(\pi x/2) = \prod_{n=0}^{\infty}\Big(1-\frac{x^2}{(2n+1)^2}\Big), \qquad \text{for } x\in\mathbb{R}. \tag{5}$$

Proposition 2.3. *For any $k,m\in\mathbb{N}$ such that $m<k$, it follows that*

$$\frac{1}{\cos(\pi m/(2k))} = \prod_{n=0}^{\infty}\frac{2nk+k}{2nk+k-m}\,\frac{2nk+k}{2nk+k+m}. \tag{6}$$

Example 2.4. Letting $k=3$ and $m=2$ in (6), we obtain $\prod_{n=0}^{\infty}\frac{6n+3}{6n+1}\frac{6n+3}{6n+5}=2$. Letting $k=2, m=1$ in (6) yields Catalan's product $\prod_{n=0}^{\infty}\frac{4n+2}{4n+1}\frac{4n+2}{4n+3}=\sqrt{2}$ [7].

The above propositions can be generalized and extended in many ways. Consider, for instance, the following example.

Example 2.5. Ramanujan [11, p. 50] showed that:

(a) $\prod_{n=1}^{\infty}\left(1+\left(\frac{2p}{n+p}\right)^3\right) = \frac{\{\Gamma(1+p)\}^3}{\Gamma(2+3p)}\,\frac{\cosh(\pi(p+1/2)\sqrt{3})}{\pi}$.

(b) $\prod_{n=1}^{\infty}\left\{\left(1+\frac{p^3}{n^3}\right)\cdot\left(1+3\left(\frac{p}{n+p+1}\right)^2\right)\right\} = \frac{\Gamma(p/2)}{\Gamma((p+1)/2)}\,\frac{\cosh(\pi p\sqrt{3})-\cosh(\pi p)}{2^{x+2}\pi p\sqrt{\pi}}$.

If $p \in \mathbb{Q}$, the above formulas yield expressions for infinite products of fractions whose periodic structure resembles that of the Wallis product

In fact, the Weierstrass factorization theorem implies that [24, Section 12.13]:

$$\prod_{n=0}^{\infty}\prod_{j=1}^{d}\frac{n+x_j}{n+y_j} = \prod_{j=1}^{d}\frac{\Gamma(y_j)}{\Gamma(x_j)}, \qquad \text{for } x_j, y_j \in \mathbb{C}, \tag{7}$$

as long as $\sum_{j=1}^{d} x_j = \sum_{j=1}^{d} y_j$ and none of y_j is a negative integer or zero. We adopt the point of view proposed in [20] and perceive (7) as a recipe for creating Wallis-type identities.

Definition 2.6. *For $\alpha > 0$, we write $\alpha \in \mathbb{W}$ if $\alpha = \prod_{n=0}^{\infty}\frac{\mathcal{P}(n)}{\mathcal{Q}(n)}$ for some polynomials $\mathcal{P}$ and $\mathcal{Q}$ with positive rational roots and common degree, that is if*

$$\alpha = \prod_{n=0}^{\infty}\prod_{j=1}^{d}\frac{n+a_j/k}{n+b_j/k} = \prod_{n=0}^{\infty}\prod_{j=1}^{d}\frac{kn+a_j}{kn+b_j}, \qquad \text{for } d, k, a_j, b_j \in \mathbb{N}. \tag{8}$$

The rightmost expression in (8) *is said to be a Wallis-type infinite product for α.*

In view of (7), the condition $\sum_{j=1}^{d} a_j = \sum_{j=1}^{d} b_j$ must hold to ensure that α is finite and non-zero. A Wallis-type product can be equivalently defined as $\prod_{n=0}^{\infty}\frac{p_n}{q_n}$, where $p_n, q_n \in \mathbb{N}$ and $p_{n+d} = p_n + k$, $q_{n+d} = q_n + k$ for some $d, k \in \mathbb{N}$ and all integers $n \geq 0$ (in particular, both p_n and q_n grow asymptotically linearly). Note that the products in Example 2.5 are *not* of the Wallis-type.

3 Some further examples of Wallis-type products

The analysis of the structure of $\mathbb{W}$ turns out to be challenging. We refer to [6, 10, 21, 22] for various aspects of this problem. In this section, we present a selected variety of examples where Wallis-type products can be evaluated explicitly and explore a few links between them.

Example 3.1. Start with $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$ ("Morrie's law" [4]), which is a special case of

$$\prod_{j=0}^{p-1}\cos(2^j x) = \frac{\sin(2^p x)}{2^p \sin x}, \qquad \text{for } p \in \mathbb{N},\ x \in \mathbb{R}. \tag{9}$$

Combining this result with (6) and taking in account that $\Gamma\big(1/2\big) = \sqrt{\pi}$, one obtains that

$$\prod_{n=0}^{\infty} \prod_{m\in\{2,4,8\}} \frac{18n+9-m}{18n+9}\,\frac{18n+9+m}{18n+9} = \frac{1}{8}. \tag{10}$$

In fact, (6) and (9) with $x = \frac{\pi}{2^p+1}$ imply

$$\prod_{n=0}^{\infty}\prod_{j=1}^{p} \frac{(2^{p+1}+2)n+2^p-2^j+1}{(2^{p+1}+2)n+2^p+1}\,\frac{(2^{p+1}+2)n+2^p+2^j+1}{(2^{p+1}+2)n+2^p+1} = \frac{1}{2^p}.$$

Example 3.2. Let $\mathcal{R}_n := \big\{j \in \mathbb{N} : 1 \le j \le n \text{ and } j \text{ is relatively prime to } n\big\}$, $n \in \mathbb{N}$. The following identity (see, for instance, [22]) illustrates a result of [18] for $\prod_{j\in\mathcal{R}_n} \Gamma(j/n)$:

$$\prod\nolimits_{n=0}^{\infty} \frac{14n+1}{14n+7}\,\frac{14n+9}{14n+7}\,\frac{14n+11}{14n+7} = \frac{\{\Gamma(7/14)\}^3}{\Gamma(1/14)\,\Gamma(9/14)\,\Gamma(11/14)} = \frac{1}{4}.$$

It is curious to note that (10) above can be also derived from the result of [18] with $n = 18$.

Example 3.3. Vieta [3, p. 53] (see also [13, Chapter 1]) showed that $\frac{2}{\pi} = \prod_{n=1}^{\infty} S_n$, where $S_1 = \sqrt{\frac{1}{2}}$ and $S_n = \sqrt{\frac{1}{2}+\frac{1}{2}S_{n-1}}$ for $n > 1$. Osler considered in [16] the "united Vieta-Wallis-like products"

$$\begin{aligned} &\frac{\sin(\pi x)}{\pi x} \\ &= \prod_{m=1}^{p} \underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos(\pi x)}}}}_{(m \text{ radicals})} \prod_{n=1}^{\infty} \frac{2^p n - x}{2^p n}\,\frac{2^p n + x}{2^p n} \end{aligned} \tag{11}$$

with $p \in \mathbb{N}\cup\{0\}$. The proof of (11) rests on (4), (9), and the identity $\cos\big(\frac{\theta}{2}\big) = \sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}$.

Example 3.4. The set $\big\{\Gamma(m/k) : m, k \in \mathbb{Z};\ k \text{ divides } 24 \text{ or } 60\big\}$ is investigated in [21]. One motivation for this study is its relevance to the problem of the evaluation of hypergeometric functions and the theory of elliptic integrals. In addition, as it is observed in [21], the topic is related to an instance of the Lang-Rohrlich conjecture [10, 22]. We, for instance, have:

$$\prod_{n=0}^{\infty} \frac{30n+9}{30n+10}\,\frac{30n+19}{30n+18} = \frac{\Gamma(1/3)\Gamma(3/5)}{\Gamma(3/10)\Gamma(19/30)} = \frac{\sqrt{\sqrt{15}+\sqrt{5+2\sqrt{5}}}}{2^{19/30}3^{1/20}5^{1/3}},$$

$$\prod_{n=0}^{\infty} \frac{12n+5}{12n+6}\,\frac{12n+9}{12n+8} = \frac{\Gamma(1/2)\Gamma(2/3)}{\Gamma(3/4)\Gamma(5/12)} = \frac{\sqrt{\sqrt{3}+1}}{2^{1/4}3^{3/8}},$$

$$\prod_{n=0}^{\infty} \Big(\frac{6n+2}{6n+4}\Big)^3 \frac{6n+5}{6n+1}\,\frac{6n+5}{6n+3} = \frac{\{\Gamma(1/3)\}^3}{\{\Gamma(2/3)\}^3}\,\frac{\Gamma(5/6)}{\Gamma(1/6)}\,\frac{\Gamma(5/6)}{\Gamma(3/6)} = 1.$$

In the same spirit, using the rather surprising result (we adopt this epithet from [6]) that

$$\frac{\Gamma(1/24)}{\Gamma(5/24)}\,\frac{\Gamma(11/24)}{\Gamma(7/24)}=\sqrt{6+\sqrt{3}},$$

one can compute $\prod_{n=0}^{\infty}\frac{24n+5}{24n+1}\,\frac{24n+7}{24n+11}$. Similar formulas can be derived from other results of [21].

Example 3.5. It is observed in [1, Section 4.4] that the key formula (7) along with the so called "standard equations" for the gamma function (translation, reflection, and multiplication) can be used to derive the following identity which is valid for any integer $k \geq 0$:

$$\prod_{n=0}^{\infty}\prod_{j=1}^{k}\frac{(2n+1)(2k+1)-2j}{(2n+1)(2k+1)-2j+1}\,\frac{(2n+1)(2k+1)+2j}{(2n+1)(2k+1)+2j-1}=\frac{1}{\sqrt{2k+1}}. \tag{12}$$

The observation that this product can be evaluated using only the standard equations is interesting in the light of Rohrlich's conjecture which, informally speaking, asserts that those are the only ones available for the values of the gamma function in rational points while the others can be obtained as their consequences (see, for instance, [10] and [22, Section 4.1]).

Example 3.6. The following "k-th order Wallis's product" is somewhat similar to (12) written as $\prod_{n=1}^{\infty}\prod_{j=1}^{k}\big(\frac{n(2k+1)+2j-1}{n(2k+1)+2j}\big)^{(-1)^n}=\frac{2^{2k}}{\sqrt{2k+1}}\binom{2k}{k}^{-1}$. Using (7) and the "standard equations", it follows that

$$\begin{aligned}
A_k &:= \prod_{n=1}^{\infty}\prod_{j=0}^{k-1}\frac{2nk}{2nk-2j-1}\,\frac{2nk}{2nk+2j+1}=\prod_{j=0}^{k-1}\Gamma\Big(\frac{2j+1}{2k}\Big)\cdot\Gamma\Big(1+\frac{2j+1}{2k}\Big)\\
&= \prod_{j=0}^{k-1}\frac{2k}{2j+1}\cdot\Big\{\Gamma\Big(1+\frac{2j+1}{2k}\Big)\Big\}^2=\frac{(2k)^k}{(2k-1)!!}\left\{\frac{\prod_{m=1}^{2k-1}\Gamma\big(1+\frac{m}{2k}\big)}{\prod_{m=1}^{k-1}\Gamma\big(1+\frac{m}{k}\big)}\right\}^2\\
&= \frac{\pi^k(2k-1)!!}{2k^k}.
\end{aligned}$$

For instance, A_1 is Wallis's product and

$$A_2=\frac{4}{1}\frac{4}{3}\frac{4}{5}\frac{4}{7}\,\frac{8}{5}\frac{8}{7}\frac{8}{9}\frac{8}{11}\,\frac{12}{9}\frac{12}{11}\frac{12}{13}\frac{12}{15}\,\frac{16}{13}\frac{16}{15}\frac{16}{17}\frac{16}{19}\cdots=\frac{3\pi^2}{8}.$$

4 Pólya's urn schemes and Wallis-type products

In this section, we present a probabilistic interpretation of Wallis-type products in terms of probabilities of realizations of Pólya's urn scheme.

We begin by recalling the classical Pólya's urn scheme [14]. Throughout the discussion, $k \geq 2$ and $p \geq 2$ are fixed integer parameters. Consider an urn containing balls of p different colors, with colors labeled by elements of $\mathcal{N}_p = \{1, \dots, p\}$. At each unit of time $n = 1, 2, \dots,$ a ball is removed from the urn and is returned back after inspection of its color together with k extra balls of the same color. Let $\mathbf{a} \in \mathbb{N}^p$ be given by $\mathbf{a} = (a_1, \dots, a_p)$, where a_i denotes the number of balls of color i initially placed in the urn. We write $|\mathbf{a}| = \sum_{i=1}^p a_i$.

Let $\omega_n \in \mathcal{N}_p$ be the color of the ball sampled in the n-th draw. We refer to the sequence $(\omega_n)_{n\in\mathbb{N}}$ as a *realization of the Pólya urn scheme.* We denote the space of realizations $\mathcal{N}_p^{\mathbb{N}}$ of the urn by Ω_p and for $\omega \in \Omega_p$ define

$$A_n(\omega) = \{\overline{\omega} \in \Omega_p : \omega_j(\overline{\omega}) = \omega_j(\omega),\ j = 1, \dots, n\}. \tag{13}$$

That is, $A_n(\omega)$ is the set of realizations whose first n coordinates coincides with those of ω. Note that $\cap_{n\in\mathbb{N}} A_n(\omega) = \{\omega\}$. Let $\mathcal{F}_n$ denote the σ-algebra generated by $\omega_1, \dots, \omega_n$ and let $\mathcal{F}_\infty$ denote the σ-algebra generated by $\cup_{n\in\mathbb{N}} \mathcal{F}_n$. Let $P_\mathbf{a}$ denote the law of the urn on $\mathcal{F}_\infty$, and let $E_\mathbf{a}$ be the corresponding expectation. Also, let

$$T_{n,i}(\omega) = \sum_{j=1}^n \mathbf{1}_{\{\omega_j(\omega)=i\}} \qquad \text{and} \qquad X_{n,i}(\omega) = \frac{a_i + kT_{n,i}(\omega)}{|\mathbf{a}| + kn}$$

denote the number of times a color i ball was drawn in the first n iterations and the fraction of color i balls after n iterations, respectively. Finally, for $n \in \mathbb{N}$ write $X_n = (X_{n,1}, \dots, X_{n,p})$, $T_n = (T_{n,1}, \dots, T_{n,p})$, and let $X_0 = \frac{\mathbf{a}}{|\mathbf{a}|}$ and $T_0 = (0, \dots, 0)$. Then for all $n \geq 0$, we have

$$P_\mathbf{a}(\omega_{n+1} = i|\mathcal{F}_n)(\omega) = X_{n,i}(\omega) = \frac{a_i + kT_{n,i}(\omega)}{|\mathbf{a}| + kn}.$$

Notice that the right-hand side depends only on the number of color i balls chosen in the first n iterations, but not the order in which they were chosen. It then follows by induction that the sequence $(\omega_n)_{n\in\mathbb{N}}$ is *exchangeable.* That is, for $n \in \mathbb{N}$ and a permutation σ of $\{1, \dots, n\}$, the distributions of $(\omega_1, \dots, \omega_n)$ and $(\omega_{\sigma(1)}, \dots, \omega_{\sigma(n)})$ coincide under $P_\mathbf{a}$. In particular,

$$P_\mathbf{a}(A_n(\omega)) = \frac{\prod_{i=1}^p \prod_{j=0}^{T_{n,i}(\omega)-1} (a_i + kj)}{\prod_{j=0}^{n-1} (|\mathbf{a}| + kj)}. \tag{14}$$

The following result is well-known, it was first established by Eggenberger and Pólya in the case $p = 2$ [5, 14].

Theorem 4.1. *For any $\mathbf{a} \in \mathcal{N}_p$, $X_n = \big(X_{1,n}, \dots, X_{p,n}\big)$ converges almost surely with respect to the measure $P_\mathbf{a}$ to a random vector $X_\infty \in \mathbb{R}^p$ with the Dirichlet density*

$$f_\mathbf{a}(x_1, \dots, x_p) = \frac{\Gamma(|\mathbf{a}|/k)}{\prod_{i=1}^p \Gamma(a_i/k)} \prod_{i=1}^p x_i^{\frac{a_i}{k}-1}, \quad \textit{where } x_i > 0 \textit{ and } \sum_{i=1}^p x_i = 1.$$

We will now derive a probabilistic interpretation of the Wallis-type product, which also yields the expressions for the product as $\lim_{n\to\infty} \frac{P_{\mathbf{b}}(A_n(\omega))}{P_{\mathbf{a}}(A_n(\omega))}$ for suitable choices of $\mathbf{a}, \mathbf{b}$, and ω. We begin with the intuitively clear observation that the events we consider are asymptotically vanishing:

Proposition 4.2. *Let $\omega \in \Omega_p$. Then $P_{\mathbf{a}}(\{\omega\}) = 0$.*

Proof of Proposition 4.2. For any $i \in \mathcal{N}_p$ and $n \in \mathbb{N}$ we have

$$P(\omega_{n+1} = i|\mathcal{F}_n) = X_{n,i} \le \frac{a_i + kn}{|\mathbf{a}| + kn} \le \frac{|\mathbf{a}| - 1 + kn}{|\mathbf{a}| + kn} = 1 - \frac{1}{|\mathbf{a}| + kn}.$$

Therefore,

$$P(A_n(\omega)) \le \prod_{j=0}^{n} \Big(1 - \frac{1}{|\mathbf{a}| + kj}\Big) \le e^{-\sum_{j=0}^{n} \frac{1}{|\mathbf{a}|+kj}} \to 0, \qquad \text{as } n \to \infty.$$

This yields the result by virtue of the identity $\{\omega\} = \cap_{n\in\mathbb{N}} A_n(\omega)$. $\square$

We proceed by defining the set of "admissible sequences."

Definition 4.3. *Let $\mathcal{A}_p \subset \Omega_p$ denote the event*

$$\Big\{\omega \in \Omega_p : T_\infty(\omega) = \lim_{n\to\infty} \frac{T_n(\omega)}{n} \text{ exists and is in } (0,1)^p\Big\}.$$

We say that ω is an admissible sequence if $\omega \in \mathcal{A}_p$.

Note that $P_{\mathbf{a}}(\mathcal{A}_p) = 1$ according to Theorem 4.1. The following is our main result.

Theorem 4.4. *Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^p$ and fix $\omega \in \mathcal{A}_p$. Then*

$$\lim_{n\to\infty} \frac{P_{\mathbf{a}}(A_n(\omega))}{P_{\mathbf{b}}(A_n(\omega))} = \frac{f_{\mathbf{a}}(X_\infty(\omega))}{f_{\mathbf{b}}(X_\infty(\omega))},$$

where X_∞ is introduced in the statement of Theorem 4.1.

The proof which is given below is a direct application of the following Stirling's approximation formula for the gamma function (see, for instance, [15, Proposition 2.1]):

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x}, \qquad \text{as } x \to \infty. \tag{15}$$

We note that the proof does not rely on either Theorem 4.1 or any of the results of the preceding sections.

Proof of Theorem 4.4. Since for $x > 0$, $\Gamma(x+1) = x\Gamma(x)$, we obtain that for any $m \in \mathbb{N}$,

$$\prod_{j=0}^{m}(x+jk) = k^{m+1}\prod_{j=0}^{m}\Big(\frac{x}{k}+j\Big) = k^{m+1}\frac{\Gamma(\frac{x}{k}+(m+1))}{\Gamma(\frac{x}{k})}.$$

Thus rewriting (14) in terms of the gamma function and using (15), we obtain, as $n \to \infty$,

$$\begin{aligned}
P_{\mathbf{a}}(A_n(\omega)) &= \frac{\prod_{i=1}^{p}\Gamma(\frac{a_i}{k}+T_{n,i})/\Gamma(\frac{a_i}{k})}{\Gamma(\frac{|\mathbf{a}|}{k}+n)/\Gamma(\frac{|\mathbf{a}|}{k})} = \frac{\Gamma(\frac{|\mathbf{a}|}{k})}{\prod_{i=1}^{p}\Gamma(\frac{\mathbf{a}_i}{k})} \times \frac{\prod_{i=1}^{p}\Gamma(\frac{a_i}{k}+T_{n,i})}{\Gamma(\frac{|\mathbf{a}|}{k}+n)} \\
&\sim \frac{\Gamma(\frac{|\mathbf{a}|}{k})}{\prod_{i=1}^{p}\Gamma(\frac{\mathbf{a}_i}{k})} \times \frac{\prod_{i=1}^{p}(\frac{a_i}{k}+T_{n,i})^{\frac{a_i}{k}+T_{n,i}-\frac{1}{2}}}{(\frac{|\mathbf{a}|}{k}+n)^{\frac{|\mathbf{a}|}{k}+n}} \\
&\sim \frac{\Gamma(\frac{|\mathbf{a}|}{k})}{\prod_{i=1}^{p}\Gamma(\frac{\mathbf{a}_i}{k})} \times \frac{\prod_{i=1}^{p}T_{n,i}^{\frac{a_i}{k}+T_{n,i}-\frac{1}{2}}\cdot e^{\frac{a_i}{k}}}{n^{\frac{|\mathbf{a}|}{k}+n-\frac{1}{2}}\cdot e^{\frac{|\mathbf{a}|}{k}}} \\
&= \frac{\Gamma(\frac{|\mathbf{a}|}{k})}{\prod_{i=1}^{p}\Gamma(\frac{\mathbf{a}_i}{k})} \frac{\prod_{i=1}^{p}T_{n,i}^{T_{n,i}-\frac{1}{2}}}{n^{n-\frac{1}{2}}}\prod_{i=1}^{p}\Big(\frac{T_{n,i}}{n}\Big)^{\frac{|a_i|}{k}}.
\end{aligned}$$

Since $\omega \in \mathcal{A}_p$, we have $\lim_{n\to\infty} T_{n,i}/n = X_{\infty,i}(\omega)$. Therefore,

$$\lim_{n\to\infty}\frac{P_{\mathbf{a}}(A_n(\omega))}{P_{\mathbf{b}}(A_n(\omega))} = \frac{\Gamma(\frac{|\mathbf{a}|}{k})/\prod_{i=1}^{p}\Gamma(\frac{\mathbf{a}_i}{k})}{\Gamma(\frac{|\mathbf{b}|}{k})/\prod_{i=1}^{p}\Gamma(\frac{\mathbf{b}_i}{k})}\prod_{i=1}^{p}X_{\infty,i}^{\frac{|a_i|}{k}-\frac{|b_i|}{k}} = \frac{f_{\mathbf{a}}(X_\infty(\omega))}{f_{\mathbf{b}}(X_\infty(\omega))}.$$

The proof of the theorem is complete. □

We remark that the result remains true if we assume that the number of balls returned in each of the schemes is different, say k balls under $P_{\mathbf{a}}$ and $k' \neq k$ balls under $P_{\mathbf{b}}$. As the proof being identical, we omit details.

Notice that the Wallis-type products of Definition 2.6 correspond to the special case $|\mathbf{a}| = |\mathbf{b}|$ of the following corollary to our main result.

Corollary 4.5. *Suppose that* $\mathbf{a}$ *and* $\mathbf{b}$ *are elements in* $\mathbb{N}^p$. *Then*

$$\lim_{n\to\infty}\Big(\prod_{j=0}^{n}\frac{|\mathbf{b}|+kj}{|\mathbf{a}|+kj} \times \prod_{j=0}^{\lfloor n/p\rfloor}\prod_{i=1}^{p}\frac{a_i+kj}{b_i+kj}\Big) = p^{\frac{|\mathbf{b}|-|\mathbf{a}|}{k}}\frac{\Gamma(\frac{|\mathbf{a}|}{k})}{\Gamma(\frac{|\mathbf{b}|}{k})}\prod_{i=1}^{p}\frac{\Gamma(\frac{b_i}{k})}{\Gamma(\frac{a_i}{k})}. \tag{16}$$

Proof of Corollary 4.5. Let $\omega = (\omega_n)_{n\in\mathbb{N}}$ be the p-periodic sequence defined by $\omega_n \equiv n \mod p$ and $\omega_n \in [1,p]$, so that

$$\omega = 1, 2, \ldots, p, 1, 2, \ldots, p, 1, 2, \ldots .$$

Clearly $\omega \in \mathcal{A}_p$. It follows then from (14) that the left-hand side of (16) is equal to

$$\lim_{n\to\infty}\Big(\frac{\prod_{i=1}^{p}\prod_{j=0}^{T_{n,i}(\omega)-1}(a_i+kj)}{\prod_{j=0}^{n-1}|\mathbf{a}|+kj}\Big/\frac{\prod_{i=1}^{p}\prod_{j=0}^{T_{n,i}(\omega)-1}(b_i+kj)}{\prod_{j=0}^{n-1}|\mathbf{b}|+kj}\Big)$$
$$=\frac{f_{\mathbf{a}}(X_\infty(\omega))}{f_{\mathbf{b}}(X_\infty(\omega))},$$

which establishes the claim since $X_\infty(\omega)=\big(\frac{1}{p},\ldots,\frac{1}{p}\big)$. $\square$

Admissible sequences are a natural extension of the cyclic ones that show up in Corollary 4.5. Note that while $P_{\mathbf{a}}(\mathcal{A}_p)=1$, there are only countably many cyclic sequences, and hence according to Proposition 4.2, their entire collection is a set of probability zero. To get a somewhat less trivial example of a subset of $\mathcal{A}_p$ which is a null-set of measure $P_{\mathbf{a}}$, one can consider, for instance, ω's in $\mathcal{A}_p$ that do not satisfy the law of iterated logarithm for Pólya's urns (an estimate on the rate of convergence of X_n to X_∞) proved in [12, p. 775].

We conclude this section with an observation that $P_{\mathbf{a}}$ and $P_{\mathbf{b}}$ are equivalent measures, namely they share the same null-events. One thus cannot determine the initial distribution of the urn scheme only by observing its realization.

Corollary 4.6. *For $\omega \in \Omega_p$, let $Z_n(\omega)=\frac{P_{\mathbf{b}}(A_n(\omega))}{P_{\mathbf{a}}(A_n(\omega))}$. Then Z_n converges both almost surely with respect to $P_{\mathbf{a}}$ and in $L^1(P_{\mathbf{a}})$ to $\frac{f_{\mathbf{b}}(X_\infty(\omega))}{f_{\mathbf{a}}(X_\infty(\omega))}$, as $n\to\infty$. In particular, $P_{\mathbf{b}}$ is absolutely continuous with respect to $P_{\mathbf{a}}$.*

Proof of Corollary 4.6. The almost sure convergence is the content of Theorem 4.4. Observe next that $E_{\mathbf{a}}(Z_n)=1$ because Z_n is a Radon-Nikodym derivative of $P_{\mathbf{b}}|_{\mathcal{F}_n}$ with respect to $P_{\mathbf{a}}|_{\mathcal{F}_n}$ (see, for instance, Section 5.3.3 and Appendix A4 in [9] for a superb introduction to the Radon-Nikodym derivative and Radon-Nikodym theorem within the context of probability theory). In fact, it follows from the definition of $A_n(\omega)$ in (13) that

$$\begin{aligned}E_{\mathbf{a}}(Z_n) &= \sum_{\omega\in\mathcal{N}_p^n} Z_n(\omega)P_{\mathbf{a}}(A_n(\omega))=\sum_{\omega\in\mathcal{N}_p^n}\frac{P_{\mathbf{b}}(A_n(\omega))}{P_{\mathbf{a}}(A_n(\omega))}P_{\mathbf{a}}(A_n(\omega))\\ &= \sum_{\omega\in\mathcal{N}_p^n}P_{\mathbf{b}}(A_n(\omega))=1.\end{aligned}$$

Furthermore, since X_∞ has density $f_{\mathbf{a}}$ under $P_{\mathbf{a}}$, it follows that

$$E_{\mathbf{a}}(Z_\infty)=\int\frac{f_{\mathbf{b}}(x)}{f_{\mathbf{a}}(x)}f_{\mathbf{a}}(x)dx=1.$$

By Vitali's convergence theorem (see, for instance, [19, p. 165] or [9, Theorem 5.5.2]), the almost sure convergence along with the convergence of the expected values imply the convergence in $L^1(P_{\mathbf{a}})$.

It remains to show that $P_{\mathbf{b}}$ is absolutely continuous with respect to $P_{\mathbf{a}}$. Let $A \in \cup_{n\in\mathbb{N}}\mathcal{F}_n$. Then $P_{\mathbf{b}}(A) = E_{\mathbf{a}}(\mathbf{1}_A Z_n)$ for all n sufficiently large. Since

$$E_{\mathbf{a}}\big(\mathbf{1}_A|Z_n - Z_\infty|\big) \le E_{\mathbf{a}}\big(|Z_n - Z_\infty|\big) \to 0 \text{ as } n \to \infty,$$

it follows that

$$P_{\mathbf{b}}(A) = E_{\mathbf{a}}(\mathbf{1}_A Z_\infty). \tag{17}$$

Note that the mapping $A \to E_{\mathbf{a}}(\mathbf{1}_A Z_\infty)$ is a probability measure on $\mathcal{F}_\infty$. Since $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ is an algebra and the events $A \in \mathcal{F}_\infty$ satisfying (17) form a monotone class, it follows from the monotone class theorem [9, Theorem 6.1.3] that (17) holds for all $A \in \sigma\big(\cup_{n\in\mathbb{N}}\mathcal{F}_n\big) = \mathcal{F}_\infty$. □

As in the case of Theorem 4.4, the result continues to hold when the number of balls returned to each urn is different, the proof being identical.

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