A RANDOM WALK ON $\mathbb{Z}$ WITH DRIFT DRIVEN BY ITS OCCUPATION TIME AT ZERO

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Abstract. We consider a nearest neighbor random walk on the one-dimensional integer lattice with drift towards the origin determined by an asymptotically vanishing function of the number of visits to zero. We show the existence of distinct regimes according to the rate of decay of the drift. In particular, when the rate is sufficiently slow, the position of the random walk, properly normalized, converges to a symmetric exponential law. In this regime, in contrast to the classical case, the range of the walk scales differently from its position.

1. Introduction

We consider a self-interacting random walk $X := (X_n)_{n\geq 0}$ on $\mathbb{Z}$ whose drift is a function of the number of times it has already visited the origin. The random variable X_n represents the position of the walker at time $n \in \mathbb{Z}_+$. We assume that $|X_{n+1} - X_n| = 1$ for all $n \geq 0$, that is X is a nearest neighbor model. Let η_0 be a positive integer and, for $n \geq 1$, let

$$\eta_n = \eta_0 + \#\{1 \leq i \leq n : X_i = 0\}. \tag{1}$$

Thus, $\eta_n - \eta_0$ describes the number of visits of the walker to the origin by time n. Let $\varepsilon := (\varepsilon_n)_{n\geq 1}$ be a sequence taking values in $[0, 1)$. For $x \in \mathbb{Z}$ and $l \in \mathbb{N}$, let $P^{\varepsilon}_{(x,l)}$ denote a measure on the nearest neighbor random walk paths defined as follows:

$$P^{\varepsilon}_{(x,l)}(X_0 = x, \eta_0 = l) = 1$$

$$P^{\varepsilon}_{(x,l)}(X_{n+1} = j | X_n = i, \eta_n = m) = \begin{cases} \frac{1}{2} & \text{if } i = 0 \text{ and } |j| = 1 \\ \frac{1}{2}\big(1 - \text{sign}(i)\varepsilon_m\big) & \text{if } i \neq 0 \text{ and } j - i = 1. \\ \frac{1}{2}\big(1 + \text{sign}(i)\varepsilon_m\big) & \text{if } i \neq 0 \text{ and } j - i = -1. \end{cases} \tag{2}$$

Here $\text{sign}(x)$ is $-1, 0,$ or 1 according to whether x is negative, zero or positive respectively. The corresponding expectation is denoted by $E^{\varepsilon}_{(x,l)}$.

To simplify the notation, we usually denote $P^{\varepsilon}_{(0,1)}$ by P and $E^{\varepsilon}_{(0,1)}$ by E. If $\varepsilon_n = 0$ for all $n \geq 1$, we denote P by $\mathbb{P}$, E by $\mathbb{E}$, and refer to X as the *simple random walk on* $\mathbb{Z}$.

We note that, unless ε is a constant sequence, X is not a Markov chain under P. However, the $(X_n, \eta_n)_{n\geq 0}$ forms a time-homogeneous Markov chain.

Let $d_n = -\text{sign}(X_n)\varepsilon_{\eta_n}$ and let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ denote the σ-algebra generated by the random walk paths up to time n. Then

$$E\big(X_{n+1} - X_n | \mathcal{F}_n\big) = d_n. \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 60F17, 60F20, 60F5.

Key words and phrases. Limit theorems, renewal theorem, regular variation, excursions of random walks, oscillating random walks, invariance principle, Kakutani's dichotomy.

I. B. and A. R. thank the probability groups respectively at UBC and at UC Irvine for the hospitality during visits in which part of this work was carried out. Most of the work on this paper was carried out while A. R. enjoyed the hospitality of the Department of Mathematics at UBC as a post-doc.

That is d_n is the local drift of the random walk at time n. Note that the drift is always toward the origin.

The aim of this paper is to prove limit theorems for the model described above in the case when $\lim_{n\to\infty} \varepsilon_n = 0$. If the convergence is fast enough, the asymptotic behavior of X is similar to that of the simple random walk. In Theorem 2.1 we show that the functional central limit theorem holds when $n\varepsilon_n \to 0$ and that P and $\mathbb{P}$ are mutually absolutely continuous if and only if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. We refer to this regime as supercritical. On the other hand, when ε_n converges to 0 slowly, the process exhibits a different limiting behavior. This case is treated in Theorems 2.5–2.7. In particular, we show that when ε is a regularly varying sequence converging to 0 and satisfying $n\varepsilon_n \to \infty$, the position of the walk X_n, properly normalized, converges in distribution to a symmetric exponential random variable. In this case, in contrast to the simple random walk, the range of the walk up to time n scales differently from X_n. We call this regime subcritical. The critical regime, which essentially corresponds to sequences satisfying $c_1 \le n\varepsilon_n \le c_2$ for some $0 < c_1 \le c_2 < \infty$, is subject of future work.

Our model was inspired from a random tree model known as invasion percolation cluster (IPC) on a regular tree. The model was discussed [1], and we now briefly describe the results relevant to our work. Let $\mathcal{T}$ be a binary tree with root e. The IPC is a subtree of $\mathcal{T}$ defined inductively as follows. Assign to the vertices of the tree IID weights distributed uniformly on $[0, 1]$. Begin with the subtree consisting of one vertex e, and at each step add to the existing subtree the adjacent vertex having the minimal weight. The limiting object is the IPC. The IPC has the following structure: there's one infinite branch known as the backbone, and from the n^{th} vertex of the backbone emerges a subcritical Galton-Watson tree with offspring distribution $\text{Bin}(2, p_n)$, where p_n is random, increasing, and is asymptotically distributed according to $\frac{1}{2}(1 - \frac{Z}{n})$, where Z is a rate-1 exponential random variable. Each of these subcritical trees could be equivalently viewed as a subcritical percolation cluster with percolation parameter p_n. Note that $\lim_{n\to\infty} p_n = \frac{1}{2}$, the critical percolation probability on $\mathcal{T}$. Thus, the IPC drives itself into criticality, behavior also known as self-organized criticality. Now it is well known [18] that there exists a one-to-one correspondence between critical/subcritical Galton-Watson trees and excursions away from 0 of a suitably chosen random walk on $\mathbb{Z}_+$. The drift of the walk is equal to the expectation of the offspring distribution minus one (and is therefore negative). Considering the subcritical trees as a sequence of excursions away from 0 of random walks on $\mathbb{Z}_+$ with negative and asymptotically vanishing drifts, naturally leads to our model. Note that the drift for the excursion corresponding to the n^{th} backbone vertex has asymptotic drift $-Z/n$. This, with the self-organized criticality of the IPC, explains our choice of name for the critical regime in our model. Drift sequences decaying slower to 0 correspond to sequences of trees which are "more" subcritical, in the sense that the expected value of their offspring distribution is smaller. This explains our naming of the subcritical regime.

Another related class of random processes are oscillating random walks, namely time-homogeneous Markov chains in $\mathbb{R}^d$ with transition function which depends on the position of the chain with respect to a fixed hyperplane, cf. [17, 8].

The model can be also interpreted as describing a gambler (Sisyphus) who learns from his experience and adopts a new strategy whenever a ruin event occurs. This paper intends to be a first step towards a more general study of random walks in $\mathbb{Z}^d$ for which the transition probabilities are updated each time the walk visits a certain set.

The paper is organized as follows. The main results are collected in Section 2. Some general facts about random walks and regular varying sequence are recalled in Section 3. The proofs are contained in Section 4 (supercritical case) and Section 5 (subcritical case).

2. Statement of main results

This section presents the main results of this paper. It is divided into two parts. The first is devoted to the supercritical case while the second one covers the results for the subcritical regime.

2.1. **Supercritical Regime.** Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$, equipped with the topology of uniform convergence on compact sets. For a sequence of random variables $Z := (Z_n)_{n\geq 0}$ and each $n \geq 0$, let $\mathcal{I}_n^Z \in C(\mathbb{R}_+, \mathbb{R})$ denote the following linear interpolation of $Z_{[nt]}$:

$$\mathcal{I}_n^Z(t) = \frac{1}{\sqrt{n}}\big(([nt]+1-nt)Z_{[nt]} + (nt-[nt])Z_{[nt]+1}\big). \tag{4}$$

Here and henceforth $[x]$ denotes the integer part of a real number x.

We say that Z satisfies the invariance principle, if the sequence of processes $\big(\mathcal{I}_n^Z(t)\big)_{t\in\mathbb{R}_+}$ converges weakly in $C(\mathbb{R}_+, \mathbb{R})$ to the standard Brownian motion, as $n \to \infty$.

Suppose that Q_1 and Q_2 are probability measures on a common measurable space. We say that Q_1 and Q_2 are equivalent if they are mutually absolutely continuous, that is $Q_1(A) = 0$ for an event A if and only if $Q_2(A) = 0$. We say that Q_1 and Q_2 are singular if there exists an event A such that $0 = Q_1(A) = 1 - Q_2(A)$. We have:

Theorem 2.1.

(i) *Assume that* $\lim_{n\to\infty} n\varepsilon_n = 0$. *Then* X *satisfies the invariance principle.*

(ii) *The probability measures* P *and* $\mathbb{P}$ *are either equivalent or singular, according to whether* $\sum_{n=1}^{\infty} \varepsilon_n$ *is finite or not.*

For the sake of comparison with the subcritical regime, we now state some consequences of this result. Let

$$M_n = \max_{i\leq n} X_i \text{ and } \widetilde{M}_n = \max_{i\leq n} |X_i|. \tag{5}$$

We have:

Corollary 2.2.

(i) *Assume that* $\lim_{n\to\infty} n\varepsilon_n = 0$. *Then* $M_n/\sqrt{n}$ *(respectively* $\widetilde{M}_n/\sqrt{n}$*) converge in distribution, as* $n \to \infty$, *to* $\sup_{0\leq t\leq 1} B_t$ *(respectively to* $\sup_{0\leq t\leq 1} |B_t|$*), where* B_t *is the standard Brownian motion.*

(ii) *Assume that* $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. *Then* $\limsup_{n\to\infty} \dfrac{X_n}{\sqrt{2n\log\log n}} = 1$, P-a.s.

This corollary extends to our model the limit theorem for the maxima and the law of the iterated logarithm of the simple random walk.

2.2. **Subcritical Regime.** First, we recall the definition of regularly varying sequences (see for example [6] or Section 1.9 of [5]).

Definition 2.3. *Let $r := (r_n)_{n\ge 1}$ be a sequence of positive reals. We say that r is* regularly varying *with index $\rho \in \mathbb{R}$, if $r_n = n^\rho \ell_n$, where $\ell := (\ell_n)_{n\ge1}$ is such that for any $\lambda > 0$, $\lim_{n\to\infty} \ell_{[\lambda n]}/\ell_n = 1$.*
The set of regularly varying sequences with index ρ is denoted by $RV(\rho)$. If $r \in RV(0)$, we say that r is slowly varying.

In this section we make the following assumption:

Assumption 2.4 (subcritical regime).
Assume that $\varepsilon \in RV(-\alpha)$ for some $\alpha \in [0,1]$. Moreover,

- *if $\alpha = 0$, assume in addition that* $\lim_{n\to\infty} \varepsilon_n = 0$;
- *if $\alpha = 1$, assume in addition that* $\lim_{n\to\infty} n\varepsilon_n/\log n = \infty$.

To state our results for this regime, we need to introduce some additional notation. We say that two sequences of real numbers $(x_n)_{n\ge1}$ and $(y_n)_{n\ge1}$ are asymptotically equivalent and write $x_n \sim y_n$ if $\lim_{n\to\infty} x_n/y_n = 1$. Let

$$T_0 = 0 \quad \text{and} \quad T_{n+1} = \inf\{k > T_n : X_k = 0\},\ n \in \mathbb{Z}_+. \tag{6}$$

That is, T_n is the time of the n^{th} return to 0. Let

$$a_n = n + \sum_{i=1}^{n} \frac{1}{\varepsilon_i}, \qquad c_n = \min\{i \in \mathbb{N} : a_i \ge n\}, \quad \text{and} \quad b_n = \frac{1}{\varepsilon_{c_n}}. \tag{7}$$

Lemma 3.1 below shows that $a_n = E(T_n)$. The sequence $(c_n)_{n\ge1}$ is an inverse of $(a_n)_{n\ge1}$, and, by a renewal theorem of Smith [20], $c_n \sim E(\eta_n)$. Therefore, b_n can be understood as a typical lifetime of the last excursion from the origin completed before time n. The sequences $(a_n)_{n\ge1}$, $(b_n)_{n\ge1}$, and $(c_n)_{n\ge1}$ are regularly varying, and their asymptotic behavior, as $n \to \infty$, can be deduced from the standard results collected in Theorem 3.4 (see Corollary 3.5). For the distinguished case $\varepsilon_n = n^{-\alpha}$ with $\alpha \in (0,1)$, we have $a_n \sim (1+\alpha)^{-1} n^{1+\alpha}$, $c_n \sim (1+\alpha)^{\frac{1}{1+\alpha}} n^{\frac{1}{1+\alpha}}$, and hence $b_n \sim (1+\alpha)^{\frac{\alpha}{1+\alpha}} n^{\frac{\alpha}{1+\alpha}}$.

We have:

Theorem 2.5. *Let Assumption 2.4 hold. Then, as $n \to \infty$, X_n/b_n converges in distribution to a random variable with density $e^{-2|x|}$, $x \in (-\infty, \infty)$.*

Due to the symmetry of the law of X, the theorem is equivalent to the statement that $|X_n|/b_n$ converges in distribution to a rate-2 exponential random variable. The proof of Theorem 2.5 is based on a comparison of the distribution of X_n to a stationary distribution of an oscillating random walk with constant drift ε_{c_n} toward the origin.

We proceed with a more precise description of X from which Theorem 2.5 can be also derived. The method we use could possibly be adapted to the non nearest neighbor setting, provided one could show in this more general setting that the number of visits to the origin is well-localized around its typical value.

Let $\mathfrak{N}^{(c)}$ denote Ito's excursion measure associated with the excursions of the Brownian motion with drift $c < 0$ above its infimum process, and let ζ denote the lifetime of an excursion above the infimum (see Section 3.3 below for details). Let

$$V_n = \max\{i \le n : X_i = 0\} \text{ and } \tau(V) = \min\{k > 0 : X_{V+k} = 0\},\ V \in \mathbb{Z}_+.$$

We have:

Theorem 2.6. *Let Assumption 2.4 hold. Then:*

(i) $\lim_{n\to\infty} b_{2n} P(X_{2n}=0) = 2$.

(ii) *For* $t>0$, $\lim\limits_{n\to\infty} b_{2n}^2 P\big(V_{2n} = 2n - 2[tb_{2n}^2]\big) = 2\mathfrak{N}^{(-1)}(\zeta > 2t)$.
In particular, $\lim_{n\to\infty} P\big((2n - V_{2n})/b_{2n}^2 \le x\big) = \int_0^x \mathfrak{N}^{(-1)}(\zeta > t)dt$ *for all* $x > 0$.

(iii) *For* $n \in \mathbb{N}$, *let* $Z_n = \big(Z_n(t)\big)_{t\in\mathbb{R}_+}$ *be a continuous process defined for* $k \in \mathbb{Z}_+$ *through*

$$Z_n\big(k\cdot b_{2n}^{-2}\big) = b_{2n}^{-1}|X_{V_{2n}+k\wedge\tau(V_{2n})}|,$$

and is linearly interpolated elsewhere.
Then, as $n\to\infty$, *the process* Z_n *converges weakly in* $C(\mathbb{R}_+,\mathbb{R})$ *to a non-negative process with the law* $\displaystyle\int_0^\infty \mathfrak{N}^{(-1)}(\,\cdot\,,\zeta > t)dt$.

Part (i) states that, similarly to the classical renewal theory (cf. [13, 14]), the probability to find the random walk at the origin at time $2n$ is asymptotically reciprocal to the expected duration of the of the last excursion away from the origin completed before that time. Part (ii) provides limit results on the law of the last visit time to the origin before a given time. It turns out that under Assumption 2.4, b_{2n}^2 is of smaller order that n (see Lemma 3.5 below). In particular and in contrast to the classical arc-sine law (cf. [11, p. 196]), $V_{2n}/2n$ converges in probability to 1. Finally, part (iii) is a limit theorem for the law of excursion away from 0 straddling time $2n$.

The next theorem concerns the asymptotic behavior of the maxima of X. Let

$$h_n := \frac{1}{2} b_n \log(c_n/b_n) = \frac{\log(\varepsilon_{c_n} c_n)}{2\varepsilon_{c_n}}. \tag{8}$$

Note that by Assumption 2.4, $\varepsilon_{c_n} c_n \to \infty$ as $n\to\infty$. Moreover, Corollary 3.5-(v) below shows that

$$\lim_{n\to\infty} \frac{\log(\varepsilon_{c_n} c_n)}{\log n} = \frac{1-\alpha}{1+\alpha}.$$

When $\varepsilon_n = n^{-\alpha}$ with $\alpha \in (0,1)$, we have $h_n \sim \frac{1}{2}(1-\alpha)(1+\alpha)^{\frac{-1+\alpha}{1+\alpha}} n^{\frac{\alpha}{1+\alpha}} \log n$ as $n\to\infty$.

Recall the random variables $M_n, \widetilde{M}_n$ defined in (5). We prove in Section 5:

Theorem 2.7. *Let Assumption 2.4 hold. Then*

$$\lim_{n\to\infty} \frac{1}{\log(\varepsilon_{c_n} c_n)} \log\big(-\log P(\widetilde{M}_n \le x h_n)\big) = 1 - x,\;\; x > 0.$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log(\varepsilon_{c_n} c_n)} \log P\big(\widetilde{M}_n > x h_n\big) = 1 - x, \qquad x \ge 1.$$

The above limits remain true when $\widetilde{M}_n$ *is replaced with* M_n.

Corollary 2.8. *Let Assumption 2.4 hold. Then*

$$\limsup_{n\to\infty} X_n/h_n = \lim_{n\to\infty} M_n/h_n = \lim_{n\to\infty} \widetilde{M}_n/h_n = 1,$$

where the limits hold P-a.s. *when* $\alpha < 1$ *and in probability when* $\alpha = 1$.

We remark that under Assumption 2.4, $\lim_{n\to\infty} h_n/b_n = \infty$, and hence $\lim_{n\to\infty} X_n/M_n = 0$ in probability. In particular, Theorem 2.5 cannot be extended to a functional CLT for a piecewise-linear interpolation of X_n/b_n in $C(\mathbb{R}_+, \mathbb{R})$.

3. Preliminaries

The goal of this section is threefold. First, in Section 3.1 we state some general facts about the measure P^ε in the case when ε is a constant sequence. Second, in Section 3.2, we recall some useful properties of regularly varying sequences (see Theorem 3.4), and then apply this theorem (see Corollary 3.5) to draw conclusions regarding a_n, b_n, and c_n defined in (7). Finally, in Section 3.3 we deal with the asymptotic behavior of a sequence of random walks with a negative drift conditioned to stay positive. Lemma 3.6 is the key to the proof of the last two parts of Theorem 2.6.

3.1. **Random walks with a negative drift and oscillating random walks.** For $\delta \in [0, 1)$, let (δ) denote the constant sequence $\delta, \delta, \ldots$. To simplify the notation we write $P_j^{(\delta)}$ for $P_{(j,1)}^{(\delta)}$, $P^{(\delta)}$ for $P_{(0,1)}^{(\delta)}$, and let $E_j^{(\delta)}$ and $E^{(\delta)}$ denote the respective expectation operators. We remark that $P^{(0)} = \mathbb{P}$ while $P^{(\delta)}$ with $\delta \in (0, 1)$ correspond to so-called oscillating random walks (cf. [17, 8]). If μ is a probability distribution on $\mathbb{Z}$, we write $P_\mu^{(\delta)}$ for the probability measure $\sum_{j\in\mathbb{Z}} \mu(j) P_j^{(\delta)}$ and let $E_\mu^{(\delta)}$ denote the corresponding expectation.

Recall T_n from (6) and set

$$\tau_n = T_n - T_{n-1}, \quad n \geq 1, \tag{9}$$

where we make the convention that $\infty - \infty = \infty$. That is, τ_n is the duration of the n^{th} excursion away from 0. In the following lemma we recall a well-known explicit expression for the moment generating function of τ_n (see for instance [13, p. 273] or [11, p. 276]). The moments of τ_n can be computed as appropriate derivatives of the generating function.

Lemma 3.1. *Let $\delta \in [0, 1)$. Then*

$$E^{(\delta)}\big(s^{\tau_1}\big) = \frac{1 - \sqrt{1 - (1 - \delta^2)s^2}}{1 - \delta} \text{ for } 0 < s < \frac{1}{\sqrt{1 - \delta^2}}.$$

In particular,

$E\big(s^{\tau_n}\big) = \dfrac{1 - \sqrt{1 - (1 - \varepsilon_n^2)s^2}}{1 - \varepsilon_n}$ *for* $0 < s < (1 - \varepsilon_n^2)^{-1/2}$.
$E(\tau_n) = 1 + \varepsilon_n^{-1}$.
$E(\tau_n^2) = 1 + \varepsilon_n^{-1} + \varepsilon_n^{-2} + \varepsilon_n^{-3}$.
$E(\tau_n^3) = 1 + \varepsilon_n^{-1} + 3\varepsilon_n^{-4} + 3\varepsilon_n^{-5}$.

For our proofs in Sections 4 and 5, we need the following monotonicity result.

Lemma 3.2. *Let $\varepsilon^{(1)} := (\varepsilon_n^{(1)})_{n\geq1}$ and $\varepsilon^{(2)} := (\varepsilon_n^{(2)})_{n\geq1}$ be two sequences such that $\varepsilon_n^{(j)} \in (0, 1)$ for $j = 1, 2$ and $n \geq 1$ and $\sup_{n\geq1} \varepsilon_n^{(2)} \leq \inf_{n\geq1} \varepsilon_n^{(1)}$. Furthermore, let $x_1, x_2 \in \mathbb{Z}_+$ be such that $x_2 - x_1 \in 2\mathbb{Z}_+$. Then there exist two processes $Y^j := (Y_n^j)_{n\geq0}$, $j = 1, 2$, defined on the same probability space, such that*

(i) *For $j = 1, 2$, Y^j has the same distribution as X under $P_{x_j}^{\varepsilon^{(j)}}$.*
(ii) *$|Y_n^1| \leq |Y_n^2|$ for all $n \geq 0$.*

Proof. Let $(U_n)_{n\geq 1}$ be an IID sequence of uniform random variables on $[0,1]$. Set $Y_0^1 = x_1, Y_0^2 = x_2$, $\eta_0^1 = \eta_0^2 = 1$, and let

$$Y_{n+1}^j = Y_n^j + 2\mathbf{I}_{\left\{U_n \geq \frac{1}{2}\left(1+\mathrm{sign}(Y_n^j)\varepsilon^j_{\eta_n^j}\right)\right\}} - 1 \quad \text{and} \quad \eta_{n+1}^j = \eta_n^j + \mathbf{I}_{\{Y_{n+1}^j=0\}}.$$

Clearly, $(Y_n^j)_{n\geq 0}$ has the same distribution as X under $P_{x_j}^{\varepsilon^j}$. Moreover, using induction, it is not hard to check that for all $n \geq 0$, $|Y_{n+1}^2| - |Y_n^2| \geq |Y_{n+1}^1| - |Y_n^1|$, unless $Y_n^1 = 0$. But, since $Y_n^2 - Y_n^1$ is an even integer, $|Y_{n+1}^1| = 1 \leq |Y_{n+1}^2|$ also in the latter case. □

In the next lemma, to avoid dealing with a periodic Markov chain, we focus on the process $(X_{2n})_{n\geq 0}$ rather than on $X = (X_n)_{n\geq 0}$ itself. It is well-known (see [8] for a closely related general result) that the law of the Markov chain X_{2n} under $P^{(\delta)}$ converges to its unique stationary distribution μ_δ. The latter is given by

$$\mu_\delta(0) = \frac{2\delta}{1+\delta}, \quad \mu_\delta(2i) = \frac{2\delta(1-\delta)}{(1+\delta)^3}\Big(\frac{1-\delta}{1+\delta}\Big)^{2(|i|-1)}, \; i \in \mathbb{Z}\setminus\{0\}. \tag{10}$$

Let $T = \inf\{n \geq 0 : X_n = 0\}$. A standard coupling construction for countable stationary Markov chains (see for instance [11, p. 315]) implies that

$$\sup_{A\subset 2\mathbb{Z}_+} |P^{(\delta)}(X_{2n} \in A) - \mu_\delta(A)| \leq P_{\mu_\delta}^{(\delta)}(T > 2n). \tag{11}$$

Estimating the right-hand side of (11) we get:

Lemma 3.3. *For all $\delta \in (0,1)$ and $n \geq 1$,*

$$\sup_{A\subset 2\mathbb{Z}_+} |P^{(\delta)}(X_{2n} \in A) - \mu_\delta(A)| \leq 2(1+\delta^2)^{-n}.$$

Proof of Lemma 3.3. By Chebyshev's inequality, for every $\lambda > 0$,

$$P_{\mu_\delta}(T > 2n) \leq e^{-2\lambda n} E_{\mu_\delta}^{(\delta)}(e^{\lambda T}). \tag{12}$$

By Lemma 3.1, for $j \in \mathbb{Z}$,

$$E_j^{(\delta)}(e^{\lambda T}) = \big[E_1^{(\delta)}(e^{\lambda T})\big]^{|j|} = \Big[\frac{1-\sqrt{1-(1-\delta^2)e^{2\lambda}}}{(1-\delta)e^\lambda}\Big]^{|j|}, \; e^{2\lambda}(1-\delta^2) < 1. \tag{13}$$

Note that under $P_1^{(\delta)}$, $T = \tau_1$ and its distribution is equal to the distribution of $\tau_1 - 1$ under $P_0^{(\delta)}$, hence the extra term e^λ in the denominator.

Choose $\lambda > 0$ such that $e^{2\lambda} = 1+\delta^2$. Clearly, $e^{2\lambda}(1-\delta^2) = (1-\delta^4) < 1$. Therefore,

$$\begin{aligned}
P_{\mu_\delta}(T > 2n) &\underset{(12)}{\leq} (1+\delta^2)^{-n} \sum_{j\in\mathbb{Z}} \mu_\delta(2j) E_{2j}^{(\delta)}(e^{\lambda T}) \\
&\underset{(10),(13)}{=} \frac{1}{(1+\delta^2)^n}\frac{2\delta}{1+\delta}\Big[1 + \frac{2(1-\delta)}{(1+\delta)^2}\sum_{j=1}^\infty \Big(\frac{1-\delta}{1+\delta}\Big)^{2(j-1)}\Big(\frac{1-\sqrt{1-(1-\delta^2)e^{2\lambda}}}{(1-\delta)e^\lambda}\Big)^{2j}\Big] \\
&= \frac{1}{(1+\delta^2)^n}\frac{2}{1+\delta} \leq 2(1+\delta^2)^{-n},
\end{aligned}$$

completing the proof. □

3.2. **Regularly varying sequences.** We next recall some fundamental properties of regularly varying sequences that are required for our proofs in the subcritical regime.

Theorem 3.4. [5], [6] *Let $r := (r_n)_{n\geq 1} \in RV(\rho)$ for some $\rho \in \mathbb{R}$.*

(i) *Suppose that $\rho > -1$. Then $\lim_{n\to\infty} \frac{1}{nr_n}\sum_{m=1}^{n} r_m = \frac{1}{1+\rho}$.*
(ii) *Suppose that $\rho \geq 0$. Let $(j_n)_{n\geq 1}$ be a sequence of integers such that $\lim_{n\to\infty} j_n/n = \gamma$ for some $\gamma \in (0,1]$. Then $\max_{j_n\leq i\leq n} r_i \sim r_n$ and $\min_{j_n\leq i\leq n} r_i \sim \gamma^\rho r_n$ as $n\to\infty$.*
(iii) *Suppose that $\rho > 0$. Let $r^{\rm inv} := (r_n^{\rm inv})_{n\geq 1}$, where $r_n^{\rm inv} = \min\{i\geq 1 : r_i \geq n\}$. Then $r^{\rm inv} \in RV(1/\rho)$ and $r^{\rm inv}_{[r_n]} \sim r_{[r_n^{\rm inv}]} \sim n$ as $n\to\infty$.*
(iv) *Suppose that $\rho = 0$. Then $\lim_{n\to\infty} \frac{\log r_n}{\log n} = 0$.*

Corollary 3.5. *Let Assumption 2.4 hold and recall $a = (a_n)_{n\in\mathbb{N}}$, $b = (b_n)_{n\in\mathbb{N}}$, and $c = (c_n)_{n\in\mathbb{N}}$ introduced in (7). We have*

(i) *$a_n \sim (1+\alpha)^{-1} n\varepsilon_n^{-1}$ as $n\to\infty$. In particular, $a \in RV(1+\alpha)$.*
(ii) *$c \in RV(1/(1+\alpha))$.*
(iii) *$b_n = \varepsilon_{c_n}^{-1} \sim (1+\alpha)n/c_n$ as $n\to\infty$. In particular, $b \in RV(\alpha/(1+\alpha))$.*
(iv) $\lim\limits_{n\to\infty} \frac{n}{b_n^2 \log b_n} = \lim\limits_{n\to\infty} \frac{c_n^2}{n\log b_n} = \infty.$
(v) $\lim_{n\to\infty} \frac{\log(c_n/b_n)}{\log n} = \frac{1-\alpha}{1+\alpha}.$

Part (i) of the corollary follows from Theorem 3.4-(i). Once this is established, part (ii) follows from Theorem 3.4-(iii). Next, claims (i) and (iii) of Theorem 3.4 imply that

$$c_n\varepsilon_{c_n}^{-1} \sim (1+\alpha)a_{c_n} \sim (1+\alpha)n, \qquad \text{as } n\to\infty,$$

which proves (iii). To see that (iv) holds true observe that part (iii) along with Assumption 2.4 imply:

$$\frac{n\varepsilon_{c_n}^2}{\log(\varepsilon_{c_n}^{-1})} \sim \frac{1}{1+\alpha}\cdot\frac{c_n\varepsilon_{c_n}}{\log(\varepsilon_{c_n}^{-1})} \to \infty \text{ as } n\to\infty.$$

Finally, (v) follows from (ii) and (iii) combined with Theorem 3.4-(iv).

3.3. **Random walks conditioned to stay positive.** We recall some features of the excursion measure of negatively drifted Brownian motion above its infimum (cf Chapter VI.8 in [19], in particular Lemma VI.55.1). Let $c \leq 0$. Let $Z = (Z_t)_{t\in\mathbb{R}_+}$ denote the canonical process on $C(\mathbb{R}_+,\mathbb{R})$. That is $Z_t(\omega) = \omega(t)$ for $\omega \in C(\mathbb{R}_+,\mathbb{R})$. We also let $\zeta = \zeta(\omega) = \inf\{t > 0 : \omega(t) = 0\}$. We assume (by enlarging the probability space) that under P, Z is one-dimensional Brownian motion with drift c, and we let $L_t = -\inf\{Z_s : s \leq t\}$. The process $\widetilde{Z} = (\tilde{Z}_t)_{t\in\mathbb{R}_+}$ defined by $\widetilde{Z}_t := Z_t + L_t$, is a recurrent diffusion process on $\mathbb{R}_+$, and its local time at 0 is L_t. In words, the process $\widetilde{Z}$ is the drifted diffusion above its running infimum. Let $\mathcal{P}_t^{(c)}(x,y)$ denote transition function of the process $\tilde{Z}$ killed when hitting 0. That is $\mathcal{P}_t^{(c)}(x,y) = P(\widetilde{Z}_t = y, \zeta > t | \widetilde{Z}_0 = x)$. Then

$$\mathcal{P}_t^{(c)}(x,y) := \frac{1}{\sqrt{2\pi t}} e^{c(y-x)-c^2t/2}\big[e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t}\big], \qquad x,y>0, t>0.$$

Let U denote the set of excursions,

$$U = \{\omega \in C(\mathbb{R}_+,\mathbb{R}_+) : \omega(0) = 0,\ \omega(t) = 0,\ t \geq \zeta(\omega)\}.$$

By Ito's excursion theory, the distribution of the excursions of $\widetilde{Z}$ away from 0, corresponds to a Poisson point process on $(0,\infty)\times U$ with intensity measure $dt\times\mathfrak{N}^{(c)}$, where $\mathfrak{N}^{(c)}$ is a σ-finite measure on U whose finite dimensional distributions are obtained as follows. Define

$$\mathcal{R}_t^{(c)}(y) := \frac{2y}{\sqrt{2\pi t^3}}\exp\Big(-\frac{(y-ct)^2}{2t}\Big), \qquad y>0, t>0. \tag{14}$$

The measure $\mathcal{R}_t^{(c)}(y)dy$ is known as the entrance law associated with $\mathfrak{N}^{(c)}$. Then, for $0<t_1<\ldots<t_m$ and $x_1,\ldots,x_m>0$,

$$\mathfrak{N}^{(c)}\{f(t_k)\in dx_k : 1\le k\le m\} = \mathcal{R}_{t_1}^{(c)}(x_1)dx_1\prod_{k=2}^{m}\mathcal{P}_{t_{k-1},t_k}^{(c)}(x_{k-1},x_k)dx_k. \tag{15}$$

We note that

$$\mathfrak{N}^{(c)}(\zeta>t) = \int_0^\infty \mathcal{R}_t^{(c)}(y)dy, \tag{16}$$

and that (15) implies the following scaling relation

$$\begin{aligned}&\mathfrak{N}^{(c)}\big(\big(|c|\cdot f(t/c^2)\big)_{t\in\mathbb{R}_+}\in\cdot\,\big) = |c|\cdot\mathfrak{N}^{(-1)}\big(\big(f(t)\big)_{t\in\mathbb{R}_+}\in\cdot\,\big), \text{ and in particular}\\ &\mathfrak{N}^{(c)}(\zeta>t) = |c|\cdot\mathfrak{N}^{(-1)}(\zeta>tc^2).\end{aligned} \tag{17}$$

For $m>0$, let $C[0,m] := \{f:[0,m]\to\mathbb{R}, f \text{ continuous}\}$, equipped with the topology of uniform convergence. Let $\pi_m : C(\mathbb{R}_+,\mathbb{R})\to C[0,m]$ be the canonical projection defined by $\pi_m\omega(t)=\omega(t)$ for $t\in[0,m]$. Let $\mathfrak{N}^{(c)}(\,\cdot\,|\zeta>t) := \frac{\mathfrak{N}^{(c)}(\,\cdot\,;\zeta>t)}{\mathfrak{N}^{(c)}(\zeta>t)}$. Brownian meander is the time-inhomogeneous continuous Markov process on $[0,1]$, whose law is $\mathfrak{M}^{(0)}$ is defined through

$$\mathfrak{M}^{(0)}(A) := \mathfrak{N}^{(0)}(\pi_1^{-1}A|\zeta>1), \quad A \text{ is a Borel subset of } C[0,1],$$

The meander is a weak limit of zero-mean random walks conditioned to stay positive (see [7, 15] and [9], and [3, 12] and references therein for further background). Its finite-dimensional distributions were first computed in [2], The Brownian meander can be understood as a Brownian motion conditioned to stay positive up to time 1.

Analogously, for $c<0$ we define the law of drifted Brownian meander with drift c, $\mathfrak{M}^{(c)}$, by letting

$$\mathfrak{M}^{(c)}(A) := \mathfrak{N}^{(c)}(\pi_1^{-1}A|\zeta>1), \quad A \text{ is a Borel subset of } C[0,1].$$

It is well known that a sequence of random walks with well-chosen asymptotically vanishing drifts converges in distribution to drifted Brownian motion (see for instance Theorem II.3.2 in [16]). Part (ii) of the following lemma asserts that such walks, when conditioned to stay positive up to the scaling time, also converge to a non-degenerate limit, which, not surprisingly, is the drifted Brownian meander. Part (iii) is then a direct consequence of this fact. Recall the notation $P^{(\delta)}$, introduced Section 3.1 , which corresponds to a constant sequence $\delta,\delta,\ldots$. Define

$$\Lambda_n = \{X_1>0,\ldots,X_n>0\}. \tag{18}$$

Lemma 3.6. *Let $(j_n)_{n\in\mathbb{N}}$ be a sequence of positive reals and $(m_n)_{n\in\mathbb{N}}$ be sequence of positive integers such that $\lim_{n\to\infty} j_n=\infty$, $\lim_{n\to\infty} j_n/j_{n+1}=1$, and $\lim_{n\to\infty}\varepsilon_{m_n}j_n=\gamma\in(0,\infty)$.*

Then,

(i) $\lim_{n\to\infty} j_n P^{(\varepsilon_{m_n})}(\Lambda_{[j_n^2]}) = \frac{1}{2}\mathfrak{N}^{(-\gamma)}(\zeta > 1)$.

(ii) *For $n \in \mathbb{N}$, let $Y_n = \big(Y_n(t)\big)_{t\in\mathbb{R}_+}$ be a continuous process for which $Y_n\big(j_n^{-2}k\big) = j_n^{-1}X_k$ whenever $k \in \mathbb{Z}_+$, and which is linearly interpolated elsewhere.*
Then the distribution of $\pi_1 Y_n$ under $P^{(\varepsilon_{m_n})}(\,\cdot\,|\Lambda_{[j_n^2]})$ converges to $\mathfrak{M}^{(-\gamma)}$.

(iii) *For $n \in \mathbb{N}$, let $\widetilde{Y}_n = \big(\widetilde{Y}_n(t)\big)_{t\in\mathbb{R}_+}$ be a continuous process for which $\widetilde{Y}_n\big(j_n^{-2}k\big) = j_n^{-1}X_{k\wedge T_1}$ whenever $k \in \mathbb{Z}_+$, and which is linearly interpolated elsewhere.*
Then the distribution of $\widetilde{Y}_n$ under $P^{(\varepsilon_{m_n})}(\,\cdot\,|\Lambda_{[j_n^2]})$ converges to $\mathfrak{N}^{(-\gamma)}(\,\cdot\,|\zeta > 1)$.

Proof. Since $P^{(\varepsilon)}(\Lambda_j)$ is a non-increasing function of j and $j_n/j_{n+1} \sim 1$ as $n \to \infty$, we can assume without loss of generality that $[j_n^2] \in 2\mathbb{Z}_+$.

The proof of the lemma is based on the fact that, as we already mentioned, the result is known for a symmetric random walk, and that we can explicitly compare the law of a nearest-neighbor drifted walk and the distribution $\mathbb{P}$ of the simple random walk.

(i) Let $x_0 = 0$ and $m, x_1, x_2, \ldots > 0$. Counting the number of steps to the right and to the left

$$P^{(\delta_n)}\big(\cap_{k=0}^{[j_n^2]-1}\{X_k = x_k\}; X_{[j_n^2]} = 2m\big) = \mathbb{P}\big(\cap_{k=0}^{[j_n^2]-1}\{X_k = x_k\}; X_{[j_n^2]} = 2m\big)\frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^m, \tag{19}$$

the extra factor $(1-\delta_n)^{-1}$ is due to the fact that the transition kernels of the random walk under $P^{(\delta_n)}$ and $\mathbb{P}$ coincide at the origin. In particular,

$$\begin{aligned}
P^{(\delta_n)}(\Lambda_{[j_n^2]}) &= \sum_{m\in\mathbb{Z}_+} \mathbb{P}\big(\Lambda_{[j_n^2]}; X_{[j_n^2]} = 2m\big)\frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^m \\
&= \frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\int_0^\infty \mathbb{P}\big(\Lambda_{[j_n^2]}; X_{[j_n^2]} = 2[v]\big)\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^{[v]} dv \\
&= \frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\int_0^\infty \mathbb{P}\big(X_{[j_n^2]} = 2[uj_n]|\Lambda_{[j_n^2]}\big) j_n \mathbb{P}(\Lambda_{[j_n^2]})\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^{[uj_n]} du, \qquad (20)
\end{aligned}$$

where in the last line we conditioned on $\Lambda_{[j_n^2]}$ and changed variables by letting $v = uj_n$. By the reflection principle (see for instance [11, p. 198]), $\mathbb{P}(\Lambda_{[j_n^2]}) = \frac{1}{2}\mathbb{P}(X_{[j_n^2]} = 0)$. By the local limit theorem, (see for instance [11, p. 199]), $\lim_{n\to\infty} j_n\mathbb{P}(X_{[j_n^2]} = 0) = \sqrt{\frac{2}{\pi}}$. Therefore $\lim_{n\to\infty} j_n\mathbb{P}(\Lambda_{[j_n^2]}) = \frac{1}{\sqrt{2\pi}}$. Furthermore (see for instance [15]), the sequence of probability measures (ν_n) defined on Borel sets $A \subset \mathbb{R}_+$ by

$$\nu_n(A) := j_n \int_A \mathbb{P}\big(X_{[j_n^2]} = 2[j_n u]\big|\Lambda_{[j_n^2]}\big) du$$

converges weakly to a dilution of the Rayleigh distribution on $\mathbb{R}_+$ with the probability density $4ue^{-2u^2}du$. Finally, observe that $\lim_{n\to\infty}\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^{[uj_n]} = e^{-2\gamma u}$, uniformly on, say, $[1,\infty]$, and $\lim_{n\to\infty} \dfrac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n} = e^{-\gamma^2/2}$, and that the integrand in (20) is uniformly bounded. It follows that

$$\lim_{n\to\infty} j_n P^{(\delta_n)}(\Lambda_{[j_n^2]}) = \frac{1}{\sqrt{2\pi}}\int_0^\infty 4ue^{-\frac{(2u-\gamma)^2}{2}}du = \frac{1}{2}\times\frac{1}{\sqrt{2\pi}}\int_0^\infty 2ye^{-\frac{(y-\gamma)^2}{2}}dy$$
$$\underset{(14)}{=} \frac{1}{2}\int_0^\infty \mathcal{R}_1^{(-\gamma)}(y)dy \underset{(16)}{=} \frac{1}{2}\mathfrak{N}^{(-\gamma)}(\zeta>1).$$

□

(ii) First we will prove the convergence of finite-dimensional distributions. It follows from (19) that for any $l\in\mathbb{N}$, positive reals $0<t_1<\cdots<t_l\le 1$, Borel sets $A_k\subset\mathbb{R}_+\backslash\{0\}$ and $k=1,\dots,l$,

(21)

$$P^{(\delta_n)}\Big(\cap_{k=1}^l\{Y_n(t_k)\in A_k\}\Big|\Lambda_{[j_n^2]}\Big)$$
$$=\frac{\sum_{m\in\mathbb{Z}_+}\mathbb{P}\Big(\cap_{k=1}^l\{Y_n(t_k)\in A_k\};X_{[j_n^2]}=2m\Big|\Lambda_{[j_n^2]}\Big)\mathbb{P}\big(\Lambda_{[j_n^2]}\big)\frac{(1-\delta_n^2)^{[j_n^2]/2}}{1-\delta_n}\Big(\frac{1-\delta_n}{1+\delta_n}\Big)^m}{P^{(\delta_n)}\big(\Lambda_{[j_n^2]}\big)}.$$

Therefore, by the central limit theorem for random walks conditioned to stay positive (see [7, 15]) combined with the first part of the lemma and the local limit theorem mentioned in the proof of part (i),

$$\lim_{n\to\infty}P^{(\delta_n)}\Big(\cap_{k=1}^l\{Y_n(t_k)\in A_k\}\Big|\Lambda_{[j_n^2]}\Big)$$
$$=\sqrt{\frac{2}{\pi}}\frac{1}{\mathfrak{N}^{(-\gamma)}(\zeta>1)}\int_0^\infty du\ \mathfrak{M}^{(0)}(\cap_{k=1}^l\{Y_{t_k}\in A_k\};Y_1\in du)\exp\Big(-u\gamma-\frac{\gamma^2}{2}\Big)$$
$$=\frac{1}{\mathfrak{N}^{(-\gamma)}(\zeta>1)}\int_0^\infty du\ \mathfrak{N}^{(0)}(\cap_{k=1}^l\{Y_{t_k}\in A_k\};Y_1\in du)\exp\Big(-u\gamma-\frac{\gamma^2}{2}\Big)$$
$$=\mathfrak{M}^{(-\gamma)}(\cap_{k=1}^l\{Y_{t_k}\in A_k\}).$$

Next, tightness of the family of discrete distributions follows from the corresponding result for the simple random walk available in Section 3 of [15], along with (21). □

(iii) We use the second part of the lemma, along with the fact that the distribution of $(Y_n(t))_{t\ge 1}$ converges to a Brownian motion with drift $-\gamma$ (see for instance [16, Theorem II.3.2]). The claim then follows immediately from the Markov property (applied at time $t=1$) under $\mathfrak{N}^{(-\gamma)}(\,\cdot\,|\zeta>1)$ (cf. [19, Section VI.48]). □

4. Supercritical Regime

This section is devoted to the proof of Theorem 2.1 and is correspondingly divided into two parts. The proof of the invariance principle for X_n given in Section 4.1 uses a decomposition representing X_n as a sum of a martingale and a drift term. It is then shown that the drift term is asymptotically small compared to the martingale, and that the martingale satisfies the invariance principle. The criterion for the equivalence of P and $\mathbb{P}$ is proved in Section 4.2 by a reduction to a similar question for the law of the sequence of independent variables τ_n defined in (9).

4.1. **Invariance principle for X_n.** The first part of the following proposition states that T_n/n^2 converges in distribution, as $n \to \infty$, to the hitting time of level 1 of the standard Brownian motion, a non-degenerate stable random variable of index 1/2. The second part is required to evaluate both the variance of the martingale term as well as the magnitude of the drift in decomposition (26) below.

Proposition 4.1. *Assume that* $\lim_{n\to\infty} n\varepsilon_n = 0$. *Then*

(i) *For* $\lambda \geq 0$, $\lim_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) = e^{-\sqrt{2\lambda}}$.

(ii) $\frac{1}{n}\sum_{i=1}^n \varepsilon_i \tau_i$ *converges to zero in probability as* $n \to \infty$.

Proof.
(i) It is well-known (see for instance [11, p. 394]) that

$$\lim_{n\to\infty} \mathbb{E}\big(e^{-\lambda T_n/n^2}\big) = \lim_{n\to\infty} \big(\mathbb{E}\big(e^{-\lambda \tau_1/n^2}\big)\big)^n = e^{-\sqrt{2\lambda}}, \ \lambda \geq 0.$$

By Lemma 3.2-(i), $E\big(e^{-\lambda T_n/n^2}\big) \geq \mathbb{E}\big(e^{-\lambda T_n/n^2}\big)$. Hence $\liminf_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) \geq e^{-\sqrt{2\lambda}}$. It remains to show that $\limsup\limits_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) \leq e^{-\sqrt{2\lambda}}$.

Let $\delta \in (0,1)$. Clearly,

$$E\big(e^{-\lambda T_n/n^2}\big) \leq \prod_{k=[\delta n]}^{n} E\big(e^{-\lambda \tau_k/n^2}\big). \tag{22}$$

Thanks to Assumption 2.4, we can take n large enough that $k\varepsilon_k \leq \delta^2/2$ for all $k \geq [\delta n]$. Then, for $k \geq [\delta n]$,

$$\varepsilon_k \leq \frac{\delta^2}{2k} \leq \frac{\delta^2}{2[\delta n]} < \frac{\delta}{n}. \tag{23}$$

Using Lemma 3.2 to estimate the product in the right-hand side of (22), we get

$$E\big(e^{-\lambda T_n/n^2}\big) \leq \Big(E^{(\delta/n)}\big(e^{-\lambda \tau_1/n^2}\big)\Big)^{(1-\delta)n}. \tag{24}$$

Next, we observe that, using Lemma 3.1,

$$\begin{aligned} E^{(\delta/n)}\big(e^{-\lambda\tau_1/n^2}\big) &= \frac{1-\sqrt{1-(1-\delta^2/n^2)e^{-2\lambda/n^2}}}{1-\delta/n} \leq \frac{1-\sqrt{1-e^{-(\delta^2+2\lambda)/n^2}}}{1-\delta/n} \\ &= \mathbb{E}\big(e^{-(\delta^2/2+\lambda)\tau_1/n^2}\big)(1-\delta/n)^{-1}. \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty} E\big(e^{-\lambda T_n/n^2}\big) &\underset{(24),\,(25)}{\leq} \limsup_{n\to\infty} \big(\mathbb{E}\big(e^{-(\delta^2/2+\lambda)\tau_1/n^2}\big)\big)^{[(1-\delta)n]}(1-\delta/n)^{-(1-\delta)n} \\ &\underset{(22)}{=} e^{-(1-\delta)\sqrt{\delta^2+2\lambda}} e^{\delta(1-\delta)}. \end{aligned}$$

Letting $\delta \to 0$ completes the proof of Proposition 4.1-(i).

(ii) Fix $\delta \in (0,1)$ and let $S_1 = \frac{1}{n}\sum_{k=1}^{[\delta n]-1} \varepsilon_k\tau_k$, $S_2 = \frac{1}{n}\sum_{k=[\delta n]}^{n} \varepsilon_k \tau_k$. As before, we assume that n is large enough that (23) holds true for all $k \geq [\delta n]$. In particular, $S_2 \leq \delta T_n/n^2$. Next,

$$P(S_1+S_2 \geq 2\sqrt{\delta}) \leq P(S_1 \geq \sqrt{\delta}) + P(S_2 \geq \sqrt{\delta}) \leq \delta^{-1/2}E(S_1) + P(T_n/n^2 \geq \delta^{-1/2}).$$

By Lemma 3.1, $E(S_1) \le \frac{1}{n}\sum_{k=1}^{[\delta n]}(1+\varepsilon_k) \le 2\delta$. Therefore,

$$P(S_1+S_2 \ge 2\sqrt{\delta}) \le 2\sqrt{\delta} + P(T_n/n^2 \ge \delta^{-1/2}).$$

By part (i), the second term goes to 0 as $n\to\infty$. Letting δ go to 0 finishes the proof. □

We are now in position to give the proof of the first part of Theorem 2.1.

Proof of Theorem 2.1-(i). Recall $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$, $d_n = \mathrm{sign}(X_n)\varepsilon_{\eta_n}$, and identity (3). Let:

$$H_n = X_n - \overline{D}_n \quad \text{with} \quad \overline{D}_n := \sum_{k=0}^{n-1} d_k. \tag{26}$$

It follows from (3) that $H := (H_n, \mathcal{F}_n)_{n\ge 0}$ is a martingale.

Let $S_n = \sum_{k=1}^{\eta_n}\varepsilon_k\tau_k$. We next prove the following estimate:

$$\lim_{n\to\infty} S_n/\sqrt{n} = 0, \text{ in probability.} \tag{27}$$

Let $\delta>0$ and $m>0$. Then,

$$\{S_n > \delta\sqrt{n}\} \subseteq \{\eta_n \ge [\sqrt{mn}]\} \cup \Big\{\sum_{k=1}^{[\sqrt{mn}]}\varepsilon_k\tau_k > \delta\sqrt{n}\Big\}.$$

Hence, by Proposition 4.1-(ii), $\limsup_{n\to\infty} P(S_n \ge \delta\sqrt{n}) \le \limsup_{n\to\infty} P(\eta_n \ge [\sqrt{mn}])$. However, $\{\eta_n \ge [\sqrt{mn}]\} = \{T_{[\sqrt{mn}]} \le n\}$. Therefore,

$$\limsup_{n\to\infty} P(S_n \ge \delta\sqrt{n}) \le \limsup_{k\to\infty} P(T_k/k^2 \le 2/m).$$

By letting $m\to\infty$, and since δ is arbitrary, (27) follows from Proposition 4.1-(i).

We next apply the martingale central limit theorem [11, pp. 412] to show that H satisfies the invariance principle. Let

$$V_n = \sum_{k=1}^{n} E\big((H_{k+1}-H_k)^2\big|\mathcal{F}_k\big) = \sum_{k=1}^{n} E\big((X_{k+1}-X_k-d_k)^2\big|\mathcal{F}_k\big),$$

Due to the fact that H has bounded increments, it is enough to verify that $\lim_{n\to\infty} V_n/n = 1$ in probability. Note that by (3)

$$V_n = \sum_{k=1}^{n}\big(1-2d_k^2+d_k^2\big) = n - \sum_{k=1}^{n} d_k^2,$$

and $\sum_{k=1}^n d_k^2 \le \sum_{k=1}^n |d_k| \le S_n$. It follows from (27) that $\lim_{n\to\infty}\sum_{k=1}^n d_k^2/n = 0$ in probability, and, consequently, the invariance principle holds for H.

In order to complete the proof, by [4, Theorem 2.1, p.11], it suffices to show that for all $m>0$ and any continuous function $\varphi : C[0,m]\to\mathbb{R}$, we have $\lim_{n\to\infty} E\big(\varphi(\mathcal{I}_n^X)\big) = \lim_{n\to\infty} E\big(\varphi(\mathcal{I}_n^{H,m})\big)$, where $\mathcal{I}_n^{H,m}(t)$ coincides with $\mathcal{I}_n^H(t)$ on $[0,m]$. Note that the limit in the right-hand side exists due to the invariance principle for H. Since φ is bounded and uniformly continuous, this will follow once we prove that

$$K_n := \max_{t\in[0,m]}\big|\mathcal{I}_n^X(t) - \mathcal{I}_n^{H,m}(t)\big| \underset{n\to\infty}{\longrightarrow} 0, \text{ in } P\text{-probability.}$$

By its definition in (4), $\mathcal{I}_n^X(t)$ (resp. $\mathcal{I}_n^{H,m}(t)$) is a convex combination of $X_{[nt]}$ and $X_{[nt]+1}$ (resp. $H_{[nt]}$ and $H_{[nt]+1}$). Since $|X_{[nt]}-X_{[nt]+1}|=1$ and $|H_{[nt]}-H_{[nt]+1}|\le 2$, it follows that

$$K_n \le \max_{t\in[0,m]} \frac{|X_{[nt]}-H_{[nt]}|+3}{\sqrt{n}} \le \max_{t\in[0,m]} \frac{S_{[nt]}+3}{\sqrt{n}} \le \frac{S_{nm}+3}{\sqrt{n}} \underset{n\to\infty}{\to} 0 \text{ in } P\text{-probability},$$

where the limit in the right-hand side is due to (27). □

4.2. **Criterion for the equivalence/singularity of P and $\mathbb{P}$.**

Proof of Theorem 2.1-(ii). So far we have not explicitly defined a sample space on which X is defined. For the purpose of this section, it would be convenient to define one. Let $\Omega=\{\gamma:\mathbb{Z}_+\to\mathbb{Z}:\gamma_0=0,|\gamma_{k+1}-\gamma_k|=1\}$ denote the space of random walk paths starting from 0. We consider $X=X(\gamma)$ as the identity mapping on Ω: $X_n(\gamma)=\gamma_n$. Then $\mathcal{F}_n=\sigma(X_0,\dots,X_n), n\in\mathbb{Z}_+$, form a filtration on Ω. Let $\mathcal{F}=\sigma(\cup_{n\ge0}\mathcal{F}_n)$. Then $\mathbb{P}$ and P are probability measures on the measurable space $(\Omega,\mathcal{F})$. Clearly, the random times previously defined in terms of X could be now viewed as random variables on $(\Omega,\mathcal{F})$. More precisely, for $\gamma\in\Omega$ we have $T_0(\gamma)=0$ and for $n\ge1$,

$$T_n(\gamma)=\min\{i>T_{n-1}(\gamma):X_i=0\} \quad \text{and} \quad \tau_n(\gamma)=T_n(\gamma)-T_{n-1}(\gamma). \tag{28}$$

Now define $\mathcal{G}_n=\mathcal{F}_{T_n}$, the σ-algebra generated by the paths of X up to time T_n, and let and $\mathcal{G}=\sigma(\cup_{n\ge0}\mathcal{G}_n)$.

Under both P and $\mathbb{P}$, $\lim_{n\to\infty}T_n=\infty$ with probability one and hence $\mathcal{G}=\mathcal{F}$ up to null-measure sets. Therefore, the measures P and $\mathbb{P}$ are equivalent if $P|_{\mathcal{G}}$ and $\mathbb{P}|_{\mathcal{G}}$, their restrictions to $\mathcal{G}$, are equivalent.

Fix $\gamma'\in\Omega$. Counting the number of the steps to the left and to the right during each excursion of the random walk from zero, we obtain

$$\begin{aligned} P(X_k=\gamma'_k,\ \forall k\le T_n) &= \prod_{k=1}^n \frac{1}{2}\Big(\frac{1}{2}(1+\varepsilon_k)\Big)^{\tau_k(\gamma')/2}\Big(\frac{1}{2}(1-\varepsilon_k)\Big)^{\tau_k(\gamma')/2-1} \\ &= 2^{-T_n(\gamma')}\prod_{k=1}^n \frac{(1-\varepsilon_k^2)^{\tau_k(\gamma')/2}}{1-\varepsilon_k}, \end{aligned} \tag{29}$$

where the difference between the powers in the right-hand side of the first line is due to the fact that from 0, the probability of going either to the right or to the left is $\frac{1}{2}$. On the other hand, $\mathbb{P}(X_k=\gamma'_k,\ \forall k\le T_n)=2^{-T_n(\gamma')}$.

Let F_n denote the Radon-Nikodym derivative of $P|_{\mathcal{F}_n}$ with respect to $\mathbb{P}|_{\mathcal{F}_n}$. That is, for all $\gamma'\in\Omega$,

$$F_n(\gamma') := \frac{P(X_k=\gamma'_k,\ \forall k\le T_n)}{\mathbb{P}(X_k=\gamma'_k,\ \forall k\le T_n)} = \prod_{k=1}^n \frac{(1-\varepsilon_k^2)^{\tau_k(\gamma')/2}}{1-\varepsilon_k}, \tag{30}$$

and set $F_\infty(\gamma')=\limsup_{n\to\infty}F_n(\gamma')$. Note that $F_n\in\mathcal{G}_n$ and hence $F_\infty\in\mathcal{G}$. By [11, Theorem 3.3, p. 242],

$P|_{\mathcal{G}}$ and $\mathbb{P}|_{\mathcal{G}}$ are equivalent if and only if $F_\infty<\infty$, $P|_{\mathcal{G}}$-almost surely;

$P|_{\mathcal{G}}$ and $\mathbb{P}|_{\mathcal{G}}$ are singular if and only if $F_\infty=\infty$, $P|_{\mathcal{G}}$-almost surely.

Identity (30) with $n=1$ shows that distribution of τ_k under P is absolutely continuous with respect to its distribution under $\mathbb{P}$, and the corresponding Radon-Nikodym derivative is $(1-\varepsilon_k)^{-1}(1-\varepsilon_k^2)^{\tau_k/2}$. Since $(\tau_k)_{k\geq 1}$ is a sequence of independent random variables under both measures, Kakutani's dichotomy theorem (see [11, p. 244]) implies that

$$F_\infty < \infty \text{ or } = \infty,\ P|_{\mathcal{G}} - \text{a.s., according to whether } \lim_{n\to\infty} \mathbb{E}\big(\sqrt{F_n}\big) > 0 \text{ or } = 0.$$

We have:

$$\mathbb{E}\big(\sqrt{F_n}\big) = \prod_{k=1}^n \mathbb{E}\Big(\frac{(1-\varepsilon_k^2)^{\tau_k/4}}{\sqrt{1-\varepsilon_k}}\Big) \underset{\text{Lemma 3.1}}{=} \prod_{k=1}^n \frac{1-\sqrt{1-(1-\varepsilon_k^2)^{1/2}}}{\sqrt{1-\varepsilon_k}}.$$

Choose any $\delta \in (0, \sqrt{1/2}-1/2)$. Since $\lim_{k\to\infty} \varepsilon_k = 0$, we have for all k large enough,

$$1-\varepsilon_k\sqrt{1/2+\delta} \leq 1-\sqrt{1-(1-\varepsilon_k^2)^{1/2}} \leq 1-\varepsilon_k\sqrt{1/2},$$

and

$$1-(1/2+\delta)\varepsilon_k \leq \sqrt{1-\varepsilon_k} \leq 1-\varepsilon_k/2.$$

In particular, $\lim_{n\to\infty}\mathbb{E}\big(\sqrt{F_n}\big) > 0$ if and only if $\sum_{k=1}^\infty \varepsilon_k < \infty$. □

5. Subcritical Regime

The goal of this section is to prove the results presented in Section 2.2. In Section 5.1 we obtain auxiliary limit theorems and large deviations estimates for η_n, the occupation time at the origin. We first prove corresponding results for T_n, and then use the correspondence between $(T_n)_{n\geq 1}$ and $(\eta_n)_{n\geq 1}$. Section 5.2 contains the proof of the limit theorem for X_n stated in Theorem 2.5. In Section 5.3 we prove the more refined result given by Theorem 2.6. Finally, Theorem 2.7 and Corollary 2.8, describing the asymptotic behavior of the range of the random walk, are proved in Section 5.4.

5.1. **Limit theorems and large deviations estimates for T_n and η_n.** Let $N(0,\sigma^2)$ denote a normal random variable with zero mean and variance σ^2. We write $X_n \Rightarrow Y$ when a sequence of random variables $(X_n)_{n\geq 1}$ converges to random variable Y in distribution. Let

$$g_n := \sqrt{E(T_n^2)-(E(T_n))^2} = \Big[\sum_{i=1}^n (\varepsilon_i^{-3}-\varepsilon_i^{-1})\Big]^{1/2}, \tag{31}$$

where the first equality is the definition of g_n while the second one follows from Lemma 3.1.

First, we prove the following limit theorem for the sequence $(T_n)_{n\geq 1}$.

Proposition 5.1. *Let Assumption 2.4 hold. Then*

$$\frac{T_n-a_n}{g_n} \Rightarrow N(0,1), \ \text{ as } n\to\infty.$$

In particular, $\lim_{n\to\infty} T_n/a_n = 1$, *where the convergence is in probability.*

Next, we derive from this proposition the following limit theorem for $(\eta_n)_{n\geq 1}$.

Proposition 5.2. *Let Assumption 2.4 hold. Then*

$$\frac{\eta_n-c_n}{\sqrt{n}} \Rightarrow N\Big(0, \frac{1+\alpha}{1+3\alpha}\Big).$$

In particular, $\lim_{n\to\infty} \eta_n/c_n = 1$, *where the convergence is in probability.*

Finally, we complement the above limit results by the following large deviation estimates.

Proposition 5.3. *Let Assumption 2.4 hold. Then, for $x > 0$,*

$$\lim_{n\to\infty} \frac{1}{n\varepsilon_n} \log P\Big(\Big|\frac{T_n}{a_n} - 1\Big| > x\Big) < 0.$$

Corollary 5.4. *Let Assumption 2.4 hold. Then, for $x > 0$,*

$$\lim_{n\to\infty} \frac{b_n^2}{n} \log P\Big(\Big|\frac{\eta_n}{c_n} - 1\Big| > x\Big) < 0.$$

We note that in both Proposition 5.3 as well as Corollary 5.4 the existence of the limit in the left-hand side is part of the claim.

Corollary 5.5. *Let Assumption 2.4 hold. Then there exists a sequence $(\theta_n)_{n\ge1}$ such that $\theta_n \in (0,1)$ for all n, $\lim_{n\to\infty} \theta_n = 0$ and*

$$\lim_{n\to\infty} \exp\Big(\frac{n}{b_n^2 \log n}\Big) \cdot P\Big(\eta_n \notin \Upsilon_n\Big) = \lim_{n\to\infty} b_n^2 P\Big(\eta_n \notin \Upsilon_n\Big) = 0,$$

where $\Upsilon_n = \{m \in \mathbb{N} : |m - c_n| \le \theta_n c_n\}$.

We remark that the estimates stated in Corollary 5.5 are not optimal and, furthermore, the second is actually implied by the first one. However, the statement in the form given above is particularly convenient for reference in the sequel.

Corollary 5.4 is deduced from Proposition 5.3 using a routine argument similar to the derivation of Proposition 5.2 from Proposition 5.1, and thus its proof will be omitted. In turn, Corollary 5.5 is an immediate consequence of Corollary 5.4 and Corollary 3.5-(iv). Indeed, these two results combined together imply that

$$\lim_{n\to\infty} \exp\Big(\frac{n}{b_n^2 \log n}\Big) \cdot P\Big(\Big|\frac{\eta_n}{c_n} - 1\Big| > x\Big) = \lim_{n\to\infty} b_n^2 P(|\eta_n/c_n - 1| > x) = 0$$

for all $x > 0$. Let $n_0 = 1$, for $p \in \mathbb{N}$ let n_p be the smallest integer greater than n_{p-1} such that $\exp\big(\frac{n}{b_n^2 \log n}\big) \cdot P\big(\big|\frac{\eta_n}{c_n} - 1\big| > 1/p\big) < 1/p$ for all $n \ge n_p$, and set $\theta_n = 1/p$ for $n = n_p, \ldots, n_{p+1} - 1$.

Proof of Proposition 5.1. Let $S_n = (T_n - a_n)/g_n$. Then $E(S_n) = 0$ and $E(S_n^2) = 1$. By Lyapunov's version of the CLT for the partial sums of independent random variables, [11, p. 121], $S_n \Rightarrow N(0,1)$ if

$$\lim_{n\to\infty} \frac{1}{g_n^3} \sum_{m=1}^{n} E\big(|\tau_m - E(\tau_m)|^3\big) = 0.$$

By Lemma 3.1, $E(\tau_m) = 1 + \varepsilon_m^{-1}$. Thus, using the fact $\varepsilon_m \in (0,1)$, we obtain

$$E\big(|\tau_m - E(\tau_m)|^3\big) \le 4E\big((\tau_m - 1)^3 + \varepsilon_m^{-3}\big) \le 4(8\varepsilon_m^{-5} + \varepsilon_m^{-3}) \le 36\varepsilon_m^{-5}.$$

Next, by Theorem 3.4-(i), as $n \to \infty$, $\sum_{m=1}^{n} \varepsilon_m^{-5} \sim (1+5\alpha)^{-1} n\varepsilon_n^{-5}$ and

$$g_n^2 \sim (1+3\alpha)^{-1} n\varepsilon_n^{-3}. \tag{32}$$

Therefore,

$$\frac{1}{g_n^3} \sum_{m=1}^{n} \varepsilon_m^{-5} \sim \frac{(1+5\alpha)^{-1} n\varepsilon_n^{-5}}{(1+3\alpha)^{-3/2} n^{3/2} \varepsilon_n^{-9/2}} = \frac{(1+3\alpha)^{3/2}}{(1+5\alpha)} \frac{1}{\sqrt{n\varepsilon_n}} \to 0, \quad \text{as } n \to \infty,$$

where we use Assumption 2.4 to obtain the last limit. This completes the proof of the weak convergence of $(T_n - a_n)/g_n$.

The convergence of T_n/a_n in probability will follow, provided that $\lim_{n\to\infty} a_n/g_n = \infty$. Using again Theorem 3.4-(i), and then Assumption 2.4, we obtain, as $n \to \infty$,

$$\frac{a_n}{g_n} \sim \frac{(1+\alpha)^{-1} n\varepsilon_n^{-1}}{(1+3\alpha)^{-1/2} n^{1/2}\varepsilon_n^{-3/2}} \sim \frac{(1+3\alpha)^{1/2}}{1+\alpha}\sqrt{n\varepsilon_n} \to \infty, \quad \text{as } n \to \infty.$$

The proof of the proposition is completed. □

Proof of Proposition 5.2. First, we observe that the second statement of the proposition follows from the first one and the fact that $\lim_{n\to\infty} c_n/\sqrt{n} = \infty$ (cf. Corollary 3.5-(iv)).

We next prove the central limit theorem for η_n. As in Proposition 5.1, let g_m denote the variance of T_m and let $\widetilde{T}_m = (T_m - a_m)/g_m$. Fix $x \in \mathbb{R}$. Then

$$\begin{aligned} P\Big(\frac{\eta_n - c_n}{\sqrt{n}} \le x\Big) &= P(\eta_n \le c_n + x\sqrt{n}) = P(T_{[c_n+x\sqrt{n}]+1} > n) \\ &= 1 - P\Big(\widetilde{T}_{[c_n+x\sqrt{n}]+1} \le \frac{n - a_{[c_n+x\sqrt{n}]+1}}{g_{[c_n+x\sqrt{n}]+1}}\Big). \end{aligned} \tag{33}$$

By Corollary 3.5-(iv), $x\sqrt{n} + c_n \sim c_n$ and so

$$g_{[c_n+x\sqrt{n}]+1} \underset{(32)}{\sim} (1+3\alpha)^{-1/2}(c_n+x\sqrt{n})^{1/2}\varepsilon^{-3/2}_{[c_n+x\sqrt{n}]} \sim \sqrt{\frac{c_n\varepsilon_{c_n}^{-3}}{1+3\alpha}} \underset{\text{Theorem 3.4-(iii)}}{\sim} \sqrt{\frac{1+\alpha}{1+3\alpha}}\sqrt{n}\varepsilon_{c_n}^{-1}.$$

This leads to

$$\frac{n - a_{[c_n+x\sqrt{n}]+1}}{g_{[c_n+x\sqrt{n}]+1}} \sim -\frac{\sum_{i=c_n}^{[c_n+x\sqrt{n}]+1}\varepsilon_i^{-1}}{\sqrt{(1+\alpha)/(1+3\alpha)}\sqrt{n}\varepsilon_{c_n}^{-1}} \underset{\text{Theorem 3.4-(ii)}}{\sim} -\sqrt{\frac{1+3\alpha}{1+\alpha}}\frac{x\sqrt{n}\cdot\varepsilon_{c_n}^{-1}}{\sqrt{n}\varepsilon_{c_n}^{-1}}.$$

Therefore

$$\lim_{n\to\infty} P\Big(\frac{\eta_n - c_n}{\sqrt{n}} \le x\Big) \underset{(33)}{=} \lim_{m\to\infty} 1 - P\Big(\widetilde{T}_m \le -x\sqrt{\frac{1+3\alpha}{1+\alpha}}\Big) \underset{\text{Proposition 5.1}}{=} P\Big(N(0, \frac{1+\alpha}{1+3\alpha}) \le x\Big).$$

□

Proof of Proposition 5.3. Let $\rho_n = \min_{k\le n}\varepsilon_k$. Theorem 3.4-(ii) implies that $\rho_n \sim \varepsilon_n$ as $n \to \infty$. Let $\lambda \in (-\infty, \frac{1}{2})$ and define $\Lambda(\lambda) = \int_0^1 \big(x^{-\alpha} - \sqrt{x^{-2\alpha} - 2\lambda}\big)dx$. We shall prove that

$$\lim_{n\to\infty} \frac{1}{n\varepsilon_n} \log E\big(e^{\lambda\rho_n^2 T_n}\big) = \Lambda(\lambda). \tag{34}$$

Once this result is established, we will deduce the proposition by applying standard Chebyshev's bounds for the tail probabilities of T_n.

To prove (34) we first observe that, by Lemma 3.1,

$$\frac{1}{n\varepsilon_n} \log E\big(e^{\lambda\rho_n^2 T_n}\big) = \frac{1}{n\varepsilon_n}\sum_{i=1}^n \log\Big(1 + \frac{\varepsilon_i - \sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big). \tag{35}$$

Fix $\delta \in (0,1)$. We next show that, when n is large enough, the contribution of the first $[\delta n]$ summands on the right-hand side of (35) is bounded by a continuous function of δ which vanishes at 0. We have

$$\Big|\frac{1}{n\varepsilon_n}\sum_{i=1}^{[\delta n]}\log\Big(1+\frac{\varepsilon_i-\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big)\Big| \le \frac{1}{n\varepsilon_n}\sum_{i=1}^{[\delta n]}\frac{(1+\varepsilon_i)\big|e^{2\rho_n^2\lambda}-1\big|}{\varepsilon_i+\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}$$

$$\le \frac{2}{n\varepsilon_n}\sum_{i=1}^{[\delta n]}\frac{\big|e^{2\rho_n^2\lambda}-1\big|}{\varepsilon_i} \le \frac{2a_{[\delta n]}}{n\varepsilon_n}\big|e^{2\rho_n^2\lambda}-1\big|.$$

Since $(a_n)_{n\ge1}\in \mathrm{RV}(1+\alpha)$, Theorem 3.4 implies that, as $n\to\infty$, $a_{[\delta n]}\sim\delta^{1+\alpha}a_n\sim\frac{\delta^{1+\alpha}}{1+\alpha}\varepsilon_n^{-1}$. Therefore,

$$\frac{2a_{[\delta n]}}{n\varepsilon_n}\big|e^{2\rho_n^2\lambda}-1\big| \underset{n\to\infty}{\sim} \frac{2\delta^{1+\alpha}\varepsilon_n^{-1}}{(1+\alpha)n\varepsilon_n}2\varepsilon_n^2\lambda=\frac{4\lambda\delta^{1+\alpha}}{1+\alpha}\underset{\delta\to0}{\longrightarrow}0\ .$$

Hence,

$$\lim_{\delta\to0}\limsup_{n\to\infty}\Big|\frac{1}{n\varepsilon_n}\sum_{i=1}^{[\delta n]}\log\Big(1+\frac{\varepsilon_i-\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big)\Big|=0.$$

Next, using elementary estimates on remainders of Taylor's series, we obtain

$$\lim_{n\to\infty}\frac{1}{n\varepsilon_n}\log E\big(e^{\lambda\rho_n^2T_n}\big)=\lim_{\delta\to0}\lim_{n\to\infty}\frac{1}{n\varepsilon_n}\sum_{i=[\delta n]}^{n}\log\Big(1+\frac{\varepsilon_i-\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}{1-\varepsilon_i}\Big)$$

$$=\lim_{\delta\to0}\lim_{n\to\infty}\frac{1}{n\varepsilon_n}\sum_{i=[\delta n]}^{n}\frac{(1+\varepsilon_i)(e^{2\rho_n^2\lambda}-1)}{\varepsilon_i+\sqrt{1-(1-\varepsilon_i^2)e^{2\rho_n^2\lambda}}}=\lim_{\delta\to0}\lim_{n\to\infty}\frac{1}{n}\sum_{i=[\delta n]}^{n}\frac{2\lambda\rho_n}{\varepsilon_i+\sqrt{\varepsilon_i^2-2\rho_n^2\lambda}}$$

$$=\lim_{\delta\to0}\lim_{n\to\infty}\frac{1}{n}\sum_{i=[\delta n]}^{n}\frac{2\lambda}{\varepsilon_i/\rho_n+\sqrt{(\varepsilon_i/\rho_n)^2-2\lambda}}=\int_0^1\frac{2\lambda}{x^{-\alpha}+\sqrt{x^{-2\alpha}-2\lambda}}\,dx=\Lambda(\lambda).$$

This completes the proof of (34).

We note that $\lim_{\lambda\to-\infty}\Lambda(\lambda)=-\infty$. In addition,

$$\Lambda'(\lambda)=\int_0^1\big(x^{-2\alpha}-2\lambda\big)^{-1/2}\,dx.$$

This function is strictly increasing and hence Λ is strictly convex. Note also that $\Lambda'(0)=\frac{1}{1+\alpha}$, $\lim_{\lambda\to-\infty}\Lambda'(\lambda)=0$, and $\lim_{\lambda\to\frac12}\Lambda'(\lambda)=\infty$.

For $z>0$, let $J_z(\lambda)=\Lambda(\lambda)-\lambda z/(1+\alpha)$. This function is convex and $J_z(0)=0$. Since $J_z'(\lambda)=\Lambda'(\lambda)-z/(1+\alpha)$, the minimum of J_z is uniquely attained at some $\lambda^*\in(-\infty,\frac12)$, and $J_z(\lambda^*)<0$ for $z\ne1$. In addition, if $z>1$, $\lambda^*>0$ and if $z<1$, $\lambda^*<0$.
By Theorem 3.4-(i), as $n\to\infty$,

$$a_n\rho_n^2\sim\frac{n\varepsilon_n^{-1}\varepsilon_n^2}{1+\alpha}=\frac{n\varepsilon_n}{1+\alpha}.$$

It follows that if $\lambda\in(0,\frac12)$ then for $x>0$

$$\frac{1}{n\varepsilon_n}\log P\big(T_n/a_n\ge1+x\big)\le\frac{1}{n\varepsilon_n}\big[\log E\big(e^{\lambda\rho_n^2T_n}\big)-\lambda a_n\rho_n^2(1+x)\big]\sim J_{1+x}(\lambda).$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n\varepsilon_n} \log P\big(T_n/a_n \geq 1+x\big) \leq \min_{0<\lambda<\frac{1}{2}} J_{1+x}(\lambda) < 0.$$

Similarly, if $\lambda < 0$ then for $x \in (0,1)$

$$\frac{1}{n\varepsilon_n} \log P\big(T_n/a_n \leq 1-x\big) \leq \frac{1}{n\varepsilon_n}\big[\log E\big(e^{\lambda \rho_n^2 T_n}\big) - \lambda a_n \rho_n^2 (1-x)\big] \sim J_{1-x}(\lambda).$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n\varepsilon_n} \log P\big(T_n/a_n \leq 1-x\big) \leq \min_{\lambda<0} J_{1-x}(\lambda) < 0.$$

Moreover, since $\lim_{\lambda\to\frac{1}{2}} \Lambda'(\lambda) = \infty$, the log-generating function $\Lambda(\lambda)$ is steep in the terminology of [10]. Therefore, by the Gärtner–Ellis theorem (cf. p. 44 in [10], see also Remark (a) following the theorem), the above upper limits are in fact the limits. The proof of Proposition 5.3 is completed. □

5.2. **Proof of Theorem 2.5.** Since the law of X is symmetric about 0, the theorem is equivalent to the claim that $\lim_{n\to\infty} P(X_n > xb_n) = e^{-2x}/2$ for all $x > 0$. Furthermore, since $\lim_{n\to\infty} b_n = \infty$ and $b_n \sim b_{n+1}$, it suffices to show that

$$\lim_{n\to\infty} P(X_{2n} > xb_{2n}) = \frac{1}{2}e^{-2x}, \quad x > 0.$$

The idea of the proof is the following. In this subcritical regime, we have seen in the beginning of the section that the number of visits to the origin by time $2n$ is very-well localized around its typical value c_{2n} (cf Proposition 5.2, Corollaries 5.4 and 5.5). From properties of regular varying sequences, this will imply that the drift at time $2n$ is also very-well localized around its typical value $\varepsilon_{c_{2n}}$. Then, by Lemma 3.2, we are able to compare our walk with oscillating walks with a drift close to $\varepsilon_{c_{2n}}$, whose stationary distribution is known. In particular, Lemma 3.3 allows us to show that the distribution of X_n is close to that stationary distribution. Let us now turn to the precise argument.

Fix $x > 0$. We begin with an upper bound for $P(X_{2n} > xb_{2n})$. Recall the definition of (θ_n) from Corollary 5.5. For $n \geq 1$, let

$$\Gamma_n = \{X_n > xb_n,\ \eta_n \leq (1+\theta_n)c_n\}.$$

We have

$$P(X_{2n} > xb_{2n}) \leq P(\Gamma_{2n}) + P(\eta_{2n} > (1+\theta_{2n})c_{2n}).$$

We proceed with an estimate of the right-hand side. By Theorem 3.4-(ii), as $n\to\infty$,

$$\xi_n := \min_{i\leq(1+\theta_n)c_n} \varepsilon_i \sim \varepsilon_{(1+\theta_n)c_n} \sim \varepsilon_{c_n}. \tag{36}$$

For $n \geq 1$ consider the sequence $\alpha_n = (\alpha_{n,k})_{k\geq1}$ defined as follows: $\alpha_{n,k} = \varepsilon_k$ for $k \leq (1+\theta_n)c_n$ and $\alpha_{n,k} = \xi_n$ for $k > (1+\theta_n)c_n$. Since on event Γ_n we have $\eta_n \leq (1+\theta_n)c_n$, it follows that $P(\Gamma_{2n}) = P^{\alpha_{2n}}(\Gamma_{2n}) \leq P^{\alpha_{2n}}(X_{2n} > xb_{2n})$. Thus,

$$P(X_{2n} > xb_{2n}) \leq P^{\alpha_{2n}}(X_{2n} > xb_{2n}) + P(\eta_{2n} > (1+\theta_{2n})c_{2n}). \tag{37}$$

Recall the notation $P^{(\delta)}$ introduced in Section 3.1 (this notation is distinct from P^{δ} and emphasizes that the sequence (δ) is constant). Since $\xi_n = \min_{k\geq1} \alpha_{n,k}$, Lemma 3.2 implies

$$P^{\alpha_{2n}}(X_{2n} > xb_{2n}) \leq P^{(\xi_{2n})}(X_{2n} > xb_{2n}) = \frac{1}{2}P^{(\xi_{2n})}(|X_{2n}| > xb_{2n})$$

A second application of Lemma 3.2 shows that

$$P^{(\xi_{2n})}(|X_{2n}|>xb_{2n}) \le P_j^{(\xi_{2n})}(|X_{2n}|>xb_{2n}),\ j\in 2\mathbb{Z}.$$

In particular,

$$P^{\alpha_{2n}}(X_{2n}>xb_{2n}) \le \frac{1}{2}\sum_{j\in 2\mathbb{Z}}\mu_{\xi_{2n}}(j)P_j^{(\xi_{2n})}(|X_{2n}|>xb_{2n}) = \frac{1}{2}\times 2\mu_{\xi_{2n}}((xb_{2n},\infty)),$$

where in the last equality we have used the symmetry of $\mu_{\xi_{2n}}$ about 0. Combining this upper bound with (37) we obtain

$$P(X_{2n}>xb_{2n}) \le \mu_{\xi_{2n}}\big((xb_{2n},\infty)\big) + P(\eta_{2n}>(1+\theta_{2n})c_{2n}) \tag{38}$$

By Proposition 5.2 the second term on the right-hand side goes to 0 as $n\to\infty$. Furthermore, (10) and (36) yield that

$$\mu_{\xi_{2n}}\big((xb_{2n},\infty)\big) \sim 2\xi_{2n}\sum_{j=[xb_{2n}/2]}^{\infty}\Big(\frac{1-\xi_{2n}}{1+\xi_{2n}}\Big)^{2(j-1)} \sim \frac{1}{2}\Big(\frac{1-\xi_{2n}}{1+\xi_{2n}}\Big)^{xb_{2n}} \underset{\rho\to 0}{\longrightarrow} \frac{1}{2}e^{-2x}. \tag{39}$$

and the upper bound

$$\limsup_{n\to\infty} P(X_{2n}>xb_{2n}) \le \frac{1}{2}e^{-2x}$$

follows.

We turn to a lower bound on $P(X_{2n}>xb_{2n})$. It follows from Corollary 3.5-(iv) that there exists a sequence $(\kappa_n)_{n\ge 1}$ taking values in $2\mathbb{Z}_+$ and satisfying

$$\lim_{n\to\infty}\kappa_n/n=0 \quad\text{and}\quad \lim_{n\to\infty}\frac{\kappa_n\varepsilon_{c_n}^2}{\log(\varepsilon_{c_n}^{-1})}=\infty. \tag{40}$$

Note that the second limit in (40) ensures that $\lim_{n\to\infty}\kappa_n=\infty$. Recall Υ_n from Corollary 5.5. By Theorem 3.4-(ii), we have, as $n\to\infty$,

$$\beta_n := \max_{m\in\Upsilon_n}\varepsilon_m \sim \varepsilon_{c_n}. \tag{41}$$

Since the function $z\to z^2/\log(z^{-1})$ is increasing on $(0,1)$, the second limit in (40) along with (41) imply that $\lim_{n\to\infty}\frac{\kappa_n\beta_n^2}{\log(\beta_n^{-1})}=\lim_{n\to\infty}\frac{\kappa_n\beta_{n-\kappa_n}^2}{\log(\beta_{n-\kappa_n}^{-1})}=\infty$. Therefore,

$$\lim_{n\to\infty}(1+\beta_n^2)^{-\kappa_n}\beta_n^{-s} = \lim_{n\to\infty}(1+\beta_{n-\kappa_n}^2)^{-\kappa_n}\beta_{n-\kappa_n}^{-s}=0 \text{ for all } s\in\mathbb{R}. \tag{42}$$

We have

$$\begin{aligned} P(X_{2n}>xb_{2n}) &= \frac{1}{2}P(|X_{2n}|>xb_{2n}) \ge \frac{1}{2}P(|X_{2n}|>xb_{2n},\eta_{2n-\kappa_{2n}}\in\Upsilon_{2n-\kappa_{2n}}) \\ &= \frac{1}{2}\sum_{m\in\Upsilon_{2n-\kappa_{2n}}} P(|X_{2n}|>xb_{2n},\eta_{2n-\kappa_{2n}}=m) \\ &= \frac{1}{2}\sum_{m\in\Upsilon_{2n-\kappa_{2n}}}\sum_{j\in 2\mathbb{Z}} E\big(\mathbf{I}_{\{\eta_{2n-\kappa_{2n}}=m,X_{2n-\kappa_{2n}}=j\}}P_{(j,m)}(|X_{\kappa_{2n}}|>xb_{2n})\big). \end{aligned} \tag{43}$$

For $j\in 2\mathbb{Z}$ and $m\in\Upsilon_{2n-\kappa_{2n}}$, Lemma 3.2 implies

$$P_{(j,m)}(|X_{\kappa_{2n}}|>xb_{2n}) \ge P_j^{(\beta_{2n-\kappa_{2n}})}(|X_{\kappa_{2n}}|>xb_{2n}) \ge P^{(\beta_{2n-\kappa_{2n}})}(|X_{\kappa_{2n}}|>xb_{2n}). \tag{44}$$

Plugging this inequality into the right-hand side of (43), we obtain

$$P(X_{2n} > xb_{2n}) \geq P(\eta_{2n-\kappa_{2n}} \in \Upsilon_{2n-\kappa_{2n}})P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}} > xb_{2n}). \tag{45}$$

By Corollary 5.5, the first factor on the right-hand side converges to 1 as $n \to \infty$. Moreover, by Lemma 3.3,

$$P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}} > xb_{2n}) \geq \mu_{\beta_{2n-\kappa_{2n}}}\big((xb_{2n}, \infty)\big) - 2(1+\beta^2_{2n-\kappa_{2n}})^{-\kappa_{2n}}.$$

The second term on the right-hand side converges to 0 due to (42). Therefore, (39) and (41) imply that

$$\liminf_{n\to\infty} P(X_{2n} > xb_{2n}) \geq \frac{1}{2}e^{-2x},$$

completing the proof. □

5.3. **Proof of Theorem 2.6.**

Proof of Theorem 2.6-(i). As in Section 5.2 , the proof once again relies on Lemma 3.3 and Corollary 5.4. We adopt notation from the proof of Theorem 2.5 above.

It follows from (38) that

$$P(X_{2n} = 0) \geq \mu_{\xi_{2n}}(0) - P(\eta_{2n} \geq (1+\theta_{2n})c_{2n}).$$

Therefore,

$$b_{2n}P(X_{2n} = 0) \underset{(10)}{\geq} \frac{2}{1+\xi_{2n}}\frac{\xi_{2n}}{\varepsilon_{c_{2n}}} - b_{2n}P(\eta_{2n} \geq (1+\theta_{2n})c_{2n}).$$

The second term on the right-hand side converges to 0 due to Corollary 5.5 while the first term converges to 2 due to (36). Hence,

$$\liminf_{n\to\infty} b_{2n}P(X_{2n} = 0) \geq 2.$$

The upper bound is obtained in a similar way. We can write

$$\begin{aligned}
P(X_{2n} = 0) = 1 - P(|X_{2n}| > 2) &\underset{(45)}{\leq} 1 - P(\eta_{2n-\kappa_{2n}} \in \Upsilon_{2n-\kappa_{2n}})P^{(\beta_{2n-\kappa_{2n}})}(|X_{\kappa_{2n}}| \geq 2) \\
&= 1 - (1 - P(\eta_{2n-\kappa_{2n}} \notin \Upsilon_{2n-\kappa_{2n}}))P^{(\beta_{2n-\kappa_{2n}})}(|X_{\kappa_{2n}}| \geq 2) \\
&\leq P^{(\beta_{2n-\kappa_{2n}})}(X_{\kappa_{2n}} = 0) + P(\eta_{2n-\kappa_{2n}} \notin \Upsilon_{2n-\kappa_{2n}}) \\
&\underset{\text{Lemma 3.3}}{\leq} \mu_{\beta_{2n-\kappa_{2n}}}(0) + 2(1+\beta^2_{2n-\kappa_{2n}})^{-\kappa_{2n}} + P(\eta_{2n-\kappa_{2n}} \notin \Upsilon_{2n-\kappa_{2n}}),
\end{aligned}$$

Therefore,

$$\begin{aligned}
b_{2n}P(X_{2n} = 0) \underset{(10)}{\leq} \frac{2}{1+\beta_{2n-\kappa_{2n}}}\frac{\beta_{2n-\kappa_{2n}}}{\varepsilon_{c_{2n}}} + 2(1+\beta^2_{2n-\kappa_{2n}})^{-\kappa_{2n}}\beta^{-1}_{2n-\kappa_{2n}}\frac{\beta_{2n-\kappa_{2n}}}{\varepsilon_{c_{2n}}} \\
+ b_{2n}P(\eta_{2n-\kappa_{2n}} \notin \Upsilon_{2n-\kappa_{2n}}).
\end{aligned}$$

The third term on the right-hand side converges to 0 due to Corollary 5.5. The second term on the right-hand side converges to 0 by (41) and (42). Finally, the first term on the right-hand side converges to 2 by (41). Hence,

$$\limsup_{n\to\infty} b_{2n}P(X_{2n} = 0) \leq 2.$$

This completes the proof of the first part of Theorem 2.6. □

Proof of Theorem 2.6-(ii). Recall Λ_n from (18). By Corollary 5.5, and using the Markov property, we obtain for $t>0$,

$$\begin{aligned}
&\liminf_{n\to\infty} b_{2n}^2 P(V_{2n}=2n-2[tb_{2n}^2])\\
&\qquad = \liminf_{n\to\infty} b_{2n}^2 \sum_{m\in\Upsilon_{2n-2[tb_{2n}^2]}} P(V_{2n}=2n-2[tb_{2n}^2], \eta_{2n-2[tb_{2n}^2]}=m)\\
&\qquad = \liminf_{n\to\infty} b_{2n}^2 \sum_{m\in\Upsilon_{2n-2[tb_{2n}^2]}} P(X_{2n-2[tb_{2n}^2]}=0, \eta_{2n-2[tb_{2n}^2]}=m)\cdot 2P^{(\varepsilon_m)}(\Lambda_{2[tb_{2n}^2]}), \qquad (46)
\end{aligned}$$

The factor 2 in the last line comes from the fact that we also want count excursions below 0. Recall (41). By Lemma 3.2,

$$\begin{aligned}
&\liminf_{n\to\infty} b_{2n}^2 P(V_{2n}=2n-2[tb_{2n}^2])\\
&\qquad \geq 2\liminf_{n\to\infty} b_{2n}^2 P\big(X_{2n-2[tb_{2n}^2]}=0, \eta_{2n-2[tb_{2n}^2]}\in \Upsilon_{2n-2[tb_{2n}^2]}\big) P^{(\beta_{2n-2[tb_{2n}^2]})}(\Lambda_{2[tb_{2n}^2]}).
\end{aligned}$$

Using again Corollary 5.5, and taking into account that $\lim_{n\to\infty} b_{2n}/b_{2n-2[tb_{2n}^2]}=1$, we get

$$\liminf_{n\to\infty} b_{2n}^2 P(V_{2n}=2n-2[tb_{2n}^2]) \geq 2\liminf_{n\to\infty} b_{2n}^2 P(X_{2n-2[tb_{2n}^2]}=0) P^{(\beta_{2n-2[tb_{2n}^2]})}(\Lambda_{2[tb_{2n}^2]}).$$

Now rewrite

$$\begin{aligned}
&2b_{2n}^2 P(X_{2n-2[tb_{2n}^2]}=0) P^{(\beta_{2n-2[tb_{2n}^2]})}(\Lambda_{2[tb_{2n}^2]})\\
&\qquad = \underbrace{2b_{2n} P(X_{2n-2[tb_{2n}^2]}=0)\times \frac{b_{2n}}{\sqrt{2[tb_{2n}^2]}}\sqrt{2[tb_{2n}^2]}\,P^{(\beta_{2n-2[tb_{2n}^2]})}(\Lambda_{2[tb_{2n}^2]})}_{(*)},
\end{aligned}$$

and it follows from the first part of the theorem and Lemma 3.6-(i) that

$$\lim_{n\to\infty}(*) = \sqrt{\frac{2}{t}}\mathfrak{N}^{(-\sqrt{2t})}(\zeta>1) \underset{(17)}{=} 2\mathfrak{N}^{(-1)}(\zeta>2t).$$

Finally, a similar argument shows that

$$\limsup_{n\to\infty} b_{2n}^2 P(V_{2n}=2n-2[tb_{2n}^2]) \leq 2\mathfrak{N}^{(-1)}(\zeta>2t).$$

This proves the first claim. To prove the second claim, note that

$$\begin{aligned}
P\Big(\frac{2n-V_{2n}}{b_{2n}^2}\leq x\Big) &= \sum_{m\in\mathbb{Z}_+} P(2n-V_{2n}=2m) = \int_0^{[xb_{2n}^2]/2+1} P(2n-V_{2n}=2[v])dv\\
&= \int_0^{[xb_{2n}^2]/(2b_{2n}^2)+1/b_{2n}^2} b_{2n}^2 P(2n-V_{2n}=2[ub_{2n}^2])du.
\end{aligned}$$

Therefore by bounded convergence and the first claim

$$\lim_{n\to\infty} P\Big(\frac{2n-V_{2n}}{b_{2n}^2}\leq x\Big) = 2\int_0^{x/2} 2\mathfrak{N}^{(-1)}(\zeta>2t) = \int_0^x \mathfrak{N}^{(-1)}(\zeta>t).$$

$\square$

Proof of Theorem 2.6-(iii). We need to show that for every bounded and continuous function $F : C(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}$, $\lim_{n\to\infty} EF(Z_n) = EF(Y)$, where $Y = (Y(s))_{s\in\mathbb{R}_+}$ is a process distributed according to the law $\int_0^\infty \mathfrak{N}^{(-1)}(\cdot,\zeta > t)dt$. A straightforward calculation using (14) and (16) shows that the latter expression is indeed a probability distribution. There is no loss of generality assuming $F \geq 0$ (otherwise we can write $F = \max(F,0) - \max(-F,0)$, a difference of two bounded, continuous and non-negative functions). To prove the statement, it is sufficient to show that for all such F,

$$\liminf_{n\to\infty} EF(Z_n) \geq EF(Y), \tag{47}$$

which in particular implies that if $M = \sup F$, then

$$M - \limsup_{n\to\infty} EF(Z_n) \geq \liminf_{n\to\infty} E\big((M-F)(Z_n)\big) \geq M - EF(Y),$$

giving $\limsup_{n\to\infty} EF(Z_n) \leq EF(Y)$.

We prove (47) through a series of reductions, allowing to localize V_{2n} and the drift of Z_n, and then applying the convergence results of Lemma 3.6.

Observe that

$$EF(Z_n) = \sum_{0\leq m\leq k\leq n} E\big(F(Z_n)|2n - V_{2n} = 2k,\ \eta_{2n} = m\big)P\big(2n - V_{2n} = k,\ \eta_{2n} = m\big).$$

Now let $\widetilde{Z}_n = \big(\widetilde{Z}_n(t)\big)_{t\in\mathbb{R}_+}$ be a continuous process for which $\widetilde{Z}_n\big(k/b_{2n}^2\big) = |X_{k\wedge T_1}|/b_{2n}$ whenever $k \in \mathbb{Z}_+$, and which is linearly interpolated elsewhere. By the Markov property,

$$E\big(F(Z_n)|2n - V_{2n} = 2k,\ \eta_{2n} = m\big) = E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2k}\big).$$

Now choose $m_n \in \Upsilon_{2n}$ such that

$$E^{(\varepsilon_{m_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) = \min_{m\in\Upsilon_{2n}} E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big).$$

We then have

$$EF(Z_n) = \sum_{0\leq m\leq k\leq n} E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2k}\big)P\big(2n - V_{2n} = 2k,\ \eta_{2n} = m\big).$$

We can write this as an integral

$$\begin{aligned}
EF(Z_n) &= \int_0^{n+1} \sum_{m\leq[v]} E\big(F(\widetilde{Z}_n)\big|\Lambda_{2[v]}\big)P\big(2n - V_{2n} = 2[v],\ \eta_{2n} = m\big)dv \\
&= \int_0^{(n+1)/b_{2n}^2} \sum_{m\leq[v]} b_{2n}^2 E^{(\varepsilon_m)}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big)P\big(2n - V_{2n} = 2[tb_{2n}^2],\ \eta_{2n} = m\big)dt \\
&\geq \int_0^{(n+1)/b_{2n}^2} b_{2n}^2 E^{(\varepsilon_{m_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big)P\big(2n - V_{2n} = 2[tb_{2n}^2]\big)
\end{aligned}$$

Hence by the first part of the theorem and Fatou's lemma,

$$\liminf_{n\to\infty} EF(Z_n) \geq \int_0^\infty \liminf_{n\to\infty} E^{(\varepsilon_{m_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big)2\mathfrak{N}^{(-1)}(\zeta > 2t)dt. \tag{48}$$

Define the process $U_{n,t} = (U_{n,t}(s))_{s\in\mathbb{R}_+}$ by letting $U_{n,t}(b_{2n}^{-2}k) := \frac{1}{\sqrt{2[tb_{2n}^2]}}|X_{k\wedge T_1}|$, for $k \in \mathbb{Z}_+$ and is linearly interpolated elsewhere, and let $T_t : C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ be the mapping

$(T_t\omega)(s) = \frac{\sqrt{2[tb_{2n}^2]}}{b_{2n}}\omega(\frac{b_{2n}^2 s}{2[tb_{2n}^2]})$. We then have $Z_n = T_t U_{n,t}$. It follows from Lemma 3.6 (with $j_n = \sqrt{2[tb_{2n}^2]}$) that

$$\lim_{n\to\infty} E^{(\varepsilon_{m_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) = E^{(\varepsilon_{m_n})}\big(F\circ T_t(U_{n,t})\big|\Lambda_{2[tb_{2n}^2]}\big) = E\big(F\circ T_t(\overline{Y})\big),$$

where $\overline{Y} = (\overline{Y}(s))_{s\in\mathbb{R}_+}$ has law $\mathfrak{N}^{(-\sqrt{2t})}(\cdot|\zeta > 1)$. It follows from (17) with $c^2 = \frac{2[tb_{2n}^2]}{b_{2n}^2}$ that $T_t(\overline{Y})$ has law $\mathfrak{N}^{(-1)}(\cdot|\zeta > \frac{2[tb_{2n}^2]}{b_{2n}^2})$. Letting $n\to\infty$ we obtain

$$\lim_{n\to\infty} E^{(\varepsilon_{m_n})}\big(F(\widetilde{Z}_n)\big|\Lambda_{2[tb_{2n}^2]}\big) = E\big(F(Y)|\zeta > 2t\big).$$

Plugging this into (48) gives the desired result. □

5.4. **Proof of Theorem 2.7 and Corollary 2.8.**

Proof of Theorem 2.7. For $i \ge 1$ let $S_i = \max_{T_{i-1}\le k<T_i}|X_k|$. For $x > 0$ let $x_n = xh_n$, where h_n is defined in the statement of the theorem. Recall $(\theta_n)_{n\ge 1}$ and Υ_n from Corollary 5.5. Fix any $x \in (0,\infty)\backslash\{1\}$, $\lambda \in (0,1)$, and assume that $n \in \mathbb{N}$ below is large enough, so that $1-\theta_n > \lambda$. Then on one hand,

$$P\big(\widetilde{M}_n \le x_n\big) = P\big(\widetilde{M}_n \le x_n, \eta_n < c_n(1-\theta_n)\big) + P\big(\widetilde{M}_n \le x_n, \eta_n \ge c_n(1-\theta_n)\big)$$

$$\le P(\eta_n \notin \Upsilon_n) + \prod_{i=[\lambda c_n]}^{[c_n(1-\theta_n)]} P\big(S_i \le x_n\big), \tag{49}$$

and on the other hand, since

$$\begin{aligned}\prod_{i=1}^{[c_n(1+\theta_n)]} P\big(S_i \le x_n\big) &= P\Big(\bigcap_{i=1}^{[c_n(1+\theta_n)]}\{S_i \le x_n\}\Big)\\ &= P\Big(\bigcap_{i=1}^{[c_n(1+\theta_n)]}\{S_i \le x_n\}, \eta_n \le c_n(1+\theta_n)\Big) + P\Big(\bigcap_{i=1}^{[c_n(1+\theta_n)]}\{S_i \le x_n\}, \eta_n \ge c_n(1+\theta_n)\Big)\\ &\le P\big(\widetilde{M}_n \le x_n\big) + P\big(\eta_n \notin \Upsilon_n\big),\end{aligned}$$

we obtain

$$P\big(\widetilde{M}_n \le x_n\big) \ge \prod_{i=1}^{[c_n(1+\theta_n)]} P\big(S_i \le x_n\big) - P\big(\eta_n \notin \Upsilon_n\big), \tag{50}$$

Observe now that

$$\lim_{n\to\infty}\frac{c_n\varepsilon_{c_n}}{n\varepsilon_{c_n}^2} \underset{\text{Theorem 3.4-}(iii)}{=} \lim_{n\to\infty}\frac{c_n}{a_{c_n}\varepsilon_{c_n}} \underset{\text{Theorem 3.4-}(i)}{=} (1+\alpha) < \infty,$$

and hence, by Corollary 5.5, for all $z > 0$,

$$\lim_{n\to\infty}\frac{P(\eta_n \notin \Upsilon_n)}{e^{-(c_n\varepsilon_{c_n})^z}} = 0. \tag{51}$$

Next, by the well-known formula for the ruin probability (see for instance [11, p. 274]),

$$P(S_i \le x_n) = 1 - \frac{\rho_i}{(1+\rho_i)^{x_n}-1}, \tag{52}$$

where $\rho_i = \dfrac{2\varepsilon_i}{1-\varepsilon_i}$. For $n \in \mathbb{N}$ let

$$\chi_n := \min_{1\le i\le(1+\theta_n)c_n} \rho_i \sim 2\varepsilon_{c_n} \quad \text{and} \quad \beta_{n,\lambda} := \max_{\lambda c_n\le i\le(1+\theta_n)c_n} \rho_i \sim 2\lambda^{-\alpha}\varepsilon_{c_n}, \tag{53}$$

where we use Theorem 3.4-(iii) to state the equivalence relations. Since the righthand side in (52) is an increasing function of ρ_i, we obtain:

$$\log \prod_{i=1}^{[(1+\theta_n)c_n]} P(S_i \le x_n) \ \ge [(1+\theta_n)c_n] \log\Big(1 - \tfrac{\chi_n}{(1+\chi_n)^{x_n}-1}\Big) \underset{n\to\infty}{\sim} -\frac{c_n\chi_n}{(1+\chi_n)^{x_n}}.$$

We next estimate the rightmost expression above. Using (53) and the definition of h_n given in the statement of Theorem 2.7, we have, as $n \to \infty$,

$$\frac{1}{\log(\varepsilon_{c_n}c_n)} \cdot \log\frac{c_n\chi_n}{(1+\chi_n)^{x_n}} \sim 1 - \frac{2x\varepsilon_{c_n}h_n}{\log(\varepsilon_{c_n}c_n)} \sim 1-x,$$

Similarly, as $n \to \infty$,

$$\log \prod_{i=[\lambda c_n]}^{[(1-\theta_n)c_n]} P(S_i \le x_n) \ \le [(1-\theta_n-\lambda)c_n] \log\left(1 - \frac{\beta_{n,\lambda}}{(1+\beta_{n,\lambda})^{x_n}-1}\right) \sim -\frac{(1-\lambda)c_n\beta_{n,\lambda}}{(1+\beta_{n,\lambda})^{x_n}},$$

and

$$\frac{1}{\log(\varepsilon_{c_n}c_n)} \cdot \log\frac{(1-\lambda)c_n\beta_{n,\lambda}}{(1+\beta_{n,\lambda})^{x_n}} \sim 1 - \frac{2x\lambda^{-\alpha}\varepsilon_{c_n}h_n}{\log(\varepsilon_{c_n}c_n)} \sim 1-x\lambda^{-\alpha},$$

Since $\lambda \in (0,1)$ is arbitrary, we conclude from (49), (50), and (51) that

$$\lim_{n\to\infty} \frac{1}{\log(c_n\varepsilon_{c_n})} \log\big(-\log P(|M_n| \le x_n)\big) = 1-x.$$

Since the right-hand side is continuous in x and the left-hand side is monotone in x, the limit also holds for $x = 1$.
Note that if $x > 1$ the equality above is equivalent to the statement $\lim_{n\to\infty} \frac{1}{\log(c_n\varepsilon_{c_n})} \log\big(P(\widetilde{M}_n > x_n)\big) = 1-x$.

To complete the proof of Theorem 2.7, observe that

$$P\big(\max_{T_{i-1}\le k<T_i} X_k \le x_n\big) = \frac{1}{2} + \frac{1}{2}P\big(S_i \le x_n\big) = 1 - \frac{1}{2}\frac{\rho_i}{(1+\rho_i)^{x_n}-1}.$$

Therefore, replacing $\widetilde{M}_n$ with M_n and S_i with $\max_{T_{i-1}\le k<T_i} X_k$ in (49) and (50), the proof given above for $\widetilde{M}_n$ goes through verbatim for M_n. □

Proof of Corollary 2.8. Theorem 2.7 implies $\lim_{n\to\infty} M_n/h_n = 1$ in probability. Furthermore, by Corollary 3.5-(iii), $\varepsilon_{c_n}c_n \in \mathrm{RV}((1-\alpha)/(1+\alpha))$. Therefore if $\alpha < 1$, Theorem 2.7 implies that for any $x > 0$ there exists a constant $z = z(x) > 0$ such that

$$P\big(|M_n - h_n| > xh_n\big) \le n^{-z}$$

for all n sufficiently large.

Fix $\gamma > 1$ and let $m_n = [\gamma^n]$. Using the Borel-Cantelli lemma, we obtain that

$$P\big(|M_{m_n} - h_{m_n}| > xh_{m_n} \text{ i.o.}\big) = 0, \qquad x > 0.$$

Therefore $\lim_{n\to\infty} M_{m_n}/h_{m_n} = 1$, P-a.s. Moreover, if $m_n \le k < m_{n+1}$,

$$\frac{M_{m_n}}{h_{m_n}}\frac{h_{m_n}}{h_k} \le \frac{M_k}{h_k} \le \frac{M_{m_{n+1}}}{h_{m_{n+1}}}\frac{h_{m_{n+1}}}{h_k}.$$

Since $\lim_{n\to\infty} m_{n+1}/m_n = \gamma$ and $(h_n)_{n\ge1} \in \mathrm{RV}(\alpha/(1+\alpha))$, Theorem 3.4-(ii) implies that

$$\gamma^{-\frac{\alpha}{1+\alpha}} \le \liminf_{k\to\infty}\frac{M_k}{h_k} \le \limsup_{k\to\infty}\frac{M_k}{h_k} \le \gamma^{\frac{\alpha}{1+\alpha}}, \qquad P-\text{a.s.}$$

Since $\gamma > 1$ is arbitrary, it follows that $\lim_{k\to\infty} M_k/h_k = 1$, P-a.s. The argument can be repeated word by word for $\widetilde{M}_n$.

Finally, if $(k_n)_{n\ge1}$ is a random sequence such that $X_{k_n} = M_n$, we have:

$$\limsup_{n\to\infty}\frac{X_n}{h_n} \ge \limsup_{n\to\infty}\frac{X_{k_n}}{h_{k_n}} = \lim_{n\to\infty}\frac{M_{k_n}}{h_{k_n}} = 1,$$

where the limits hold P-a.s. when $\alpha < 1$ and in probability when $\alpha = 1$. Since $X_n \le M_n$, this completes the proof. □

Acknowledgments

We would like to express our gratitude to all the people with whom we were discussing various aspects of this paper. We wish to especially thank David Brydges and Emmanuel Jakob with whom the second and the third named authors got used to share their ideas during the work on this project. We also thank an anonymous referee for very valuable comments.

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