

Multivariate linear recursions with Markov-dependent coefficients[☆]

Diana Hay¹, Reza Rastegar¹, Alexander Roitershtein^{1,*}

Abstract

We study a linear recursion with random Markov-dependent coefficients. In a “regular variation in, regular variation out” setup we show that its stationary solution has a multivariate regularly varying distribution. This extends results previously established for i.i.d. coefficients.

Keywords: random vector equations, multivariate random recursions, stochastic difference equation, tail asymptotic, heavy tails, multivariate regular variation.

2000 MSC: Primary: 60H25, 60K15, Secondary: 60J10, 60J20.

1. Introduction

Let \mathbf{Q}_n be random d -vectors, M_n random $d \times d$ matrices, and consider the recursion

$$\mathbf{X}_n = \mathbf{Q}_n + M_n \mathbf{X}_{n-1}, \quad \mathbf{X}_n \in \mathbb{R}^d, n \in \mathbb{Z}. \quad (1)$$

This equation has been used to model the progression of real-world systems in discrete time, for example, in queuing theory [1] and financial models [2, 3]. See for instance [4, 5, 6, 7] and references therein for more examples.

Let I denote the $d \times d$ identity matrix and let $\Pi_n = M_0 M_{-1} \cdots M_{-n}$ for $n \geq 0$. It is well known (see for instance [8]) that if the sequence $(\mathbf{Q}_n, M_n)_{n \in \mathbb{Z}}$ is stationary and ergodic, and Assumption 1 stated below is imposed, then for any \mathbf{X}_0 series \mathbf{X}_n converges in distribution, as $n \rightarrow \infty$, to the random equilibrium

$$\mathbf{X} = \mathbf{Q}_0 + \sum_{k=1}^{\infty} \Pi_{-k+1} \mathbf{Q}_{-k},$$

which is the unique initial value making $(\mathbf{X}_n)_{n \geq 0}$ into a stationary sequence.

The stationary solution \mathbf{X} of the stochastic difference equation (1) has been studied by many authors. Assuming the existence of a certain “critical exponent” for M_n , the distribution tails $P(\mathbf{X} \cdot \mathbf{y} > t)$ and $P(\mathbf{X} \cdot \mathbf{y} < -t)$ for a deterministic vector $\mathbf{y} \in \mathbb{R}^d$ were shown to be regularly varied (in fact, power tailed) in [9] (for $d = 1$ an alternative

[☆]Submitted September 21, 2009; Revised June 27, 2010

*Corresponding author. E-mail: roiterst@iastate.edu

¹Department of Mathematics, Iowa State University, Ames, IA 50011, USA

proof is given in [10]). Under different assumptions and for $d = 1$ only, similar results for the tails of \mathbf{X} were obtained in [11, 12]. The multivariate recursion (1) and tails of its stationary solution \mathbf{X} were studied in [13, 14, 15] under conditions similar to those of [9], and in [16, 17] extending the one-dimensional setup of [11, 12]. In all the works mentioned above, it is assumed that $(\mathbf{Q}_n, M_n)_{n \in \mathbb{Z}}$ is an i.i.d. sequence, and [16, 17] suppose in addition that the sequences $(\mathbf{Q}_n)_{n \in \mathbb{Z}}$ and $(M_n)_{n \in \mathbb{Z}}$ are mutually independent.

The goal of this paper is to extend the results of [11, 12] to the case where $(Q_n, M_n)_{n \in \mathbb{Z}}$ are induced by a Markov chain. The extension is desirable in many, especially financial, applications, see for instance [18, 19, 20]. We remark that in dimension one the results of [9, 10] (where M_n is dominant in determining the tail behavior of \mathbf{X}) and [11, 12] (where \mathbf{Q}_n is dominant) were extended to a Markovian setup in [21, 22] and [23], respectively.

2. The setup

For $\mathbf{Q} \in \mathbb{R}^d$ define $\|\mathbf{Q}\| = \max_{1 \leq i \leq d} |\mathbf{Q}(i)|$ and let $\|M\| = \sup_{\mathbf{Q} \in \mathbb{R}^d, \|\mathbf{Q}\|=1} \|M\mathbf{Q}\|$ denote the corresponding operator norm for a $d \times d$ matrix M . The following condition ensures the existence and the uniqueness of the stationary solution to (1). The condition is also known to be close to necessity (see [24]).

Assumption 1.

(A1) $E(\log^+ \|M_0\|) < +\infty$ and $E(\log^+ \|\mathbf{Q}_0\|) < +\infty$, where $x^+ := \max\{x, 0\}$ for $x \in \mathbb{R}$.

(A2) The top Lyapunov exponent $\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|M_1 M_2 \cdots M_n\|$ is strictly negative.

Let \mathbf{I}_A denote the indicator function of the set A , that is \mathbf{I}_A is one or zero according to whether the event A occurs or not.

Definition 1. The coefficients $(\mathbf{Q}_n, M_n)_{n \in \mathbb{Z}}$ are said to be induced by a sequence of random variables $(Z_n)_{n \in \mathbb{Z}}$, each valued in a finite set \mathcal{D} , if there exists a sequence of independent random pairs $(\mathbf{Q}_{n,i}, M_{n,i})_{n \in \mathbb{Z}, i \in \mathcal{D}}$ with $\mathbf{Q}_{n,i} \in \mathbb{R}^d$ and $M_{n,i}$ being $d \times d$ matrices, such that for a fixed $i \in \mathcal{D}$, $(\mathbf{Q}_{n,i}, M_{n,i})_{n \in \mathbb{Z}}$ are i.i.d and

$$\mathbf{Q}_n = \sum_{j \in \mathcal{D}} \mathbf{Q}_{n,j} \mathbf{I}_{\{Z_n=j\}} = \mathbf{Q}_{n,Z_n} \quad \text{and} \quad M_n = \sum_{j \in \mathcal{D}} M_{n,j} \mathbf{I}_{\{Z_n=j\}} = M_{n,Z_n}. \quad (2)$$

Notice that the randomness of the coefficients $(\mathbf{Q}_n)_{n \in \mathbb{Z}}$ induced by a sequence $(Z_n)_{n \in \mathbb{Z}}$ is due to two factors:

- 1) to the randomness of the underlying auxiliary process $(Z_n)_{n \in \mathbb{Z}}$, which can be thought as representative of the “state of the external world,”

and, given the value of Z_n ,

- 2) to the “intrinsic” randomness of characteristics of the system which is captured by the random pairs $(\mathbf{Q}_{n,Z_n}, M_{n,Z_n})$.

The independence of $\mathbf{Q}_{n,i}$ and $M_{n,i}$ is not supposed in the above definition. Note that when $(Z_n)_{n \in \mathbb{Z}}$ is a finite Markov chain, (2) defines a *Hidden Markov Model* (HMM). See for instance [25] for a survey of HMM and their applications in various areas.

We will further assume that the vectors $\mathbf{Q}_{n,i}$ are *multivariate regularly varying*. Heavy tailed HMM have been considered for instance in [26], see also references therein. Recall that, for $\alpha \in \mathbb{R}$, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *regularly varying of index α* if $f(t) = t^\alpha L(t)$ for some $L(t) : \mathbb{R} \rightarrow \mathbb{R}$ such that $L(\lambda t) \sim L(t)$ for all $\lambda > 0$ (that is $L(t)$ is *slowly varying*). Here and henceforth $f(t) \sim g(t)$ (we will omit “ $t \rightarrow \infty$ ” as a rule) means $\lim_{t \rightarrow \infty} f(t)/g(t) = 1$.

Let S^{d-1} denote the unit sphere in \mathbb{R}^d with respect to the norm $\|\cdot\|$.

Definition 2. A random vector $\mathbf{Q} \in \mathbb{R}^d$ is said to be *regularly varying with index $\alpha > 0$* if there exist a function $\mathbf{a} : \mathbb{R} \rightarrow \mathbb{R}$ regularly varying with index $1/\alpha$ and a finite Borel measure $\mathfrak{S}_{\mathbf{Q}}$ on S^{d-1} such that for all $t > 0$,

$$nP(\|\mathbf{Q}\| > ta_n; \mathbf{Q}/\|\mathbf{Q}\| \in \cdot) \xrightarrow{v}_{n \rightarrow \infty} t^{-\alpha} \mathfrak{S}_{\mathbf{Q}}(\cdot), \quad \text{as } n \rightarrow \infty, \quad (3)$$

where \xrightarrow{v} denotes the vague convergence on S^{d-1} and $a_n := \mathbf{a}(n)$.

We denote by $\mathcal{R}_{d,\alpha,\mathbf{a}}$ the set of all d -vectors regularly varying with index α , associated with function \mathbf{a} by (3).

Let E be a locally compact Hausdorff topological space. The vague convergence of measures $\nu_n \xrightarrow{v}_{n \rightarrow \infty} \nu$ for finite measures ν_n , $n \geq 0$, and ν on E means (see for instance Proposition 3.12 in [27]) that $\limsup_{n \rightarrow \infty} \nu_n(K) \leq \nu(K)$ for all compact $K \subset E$ and $\liminf_{n \rightarrow \infty} \nu_n(G) \geq \nu(G)$ for all relatively compact open sets $G \subset E$. In this paper we consider vague convergence on either S^{d-1} or $\overline{\mathbb{R}}_0^d := [-\infty, \infty]^d \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ stands for the zero vector in \mathbb{R}^d . In both spaces the topology is inherited from \mathbb{R}^d (in the case of $\overline{\mathbb{R}}_0^d$ by adding neighborhoods of infinity and removing neighborhoods of zero, see for instance [28] for more details) and can be defined using an appropriate metric making both into a locally compact Polish (complete separable metric) space. A set $K \subset \overline{\mathbb{R}}_0^d$ is relatively compact if its closure does not include $\mathbf{0}$, which makes the space $\overline{\mathbb{R}}_0^d$ especially useful when convergence of regularly varying distributions is considered.

The definition (3) is norm-independent and turns out to be equivalent to the following condition (see for instance [28, 29] or [30]):

There is a Radon measure ν on $\overline{\mathbb{R}}_0^d$ such that $nP(a_n^{-1}\mathbf{Q} \in \cdot) \xrightarrow{v}_{n \rightarrow \infty} \nu(\cdot)$. The measure ν is referred to as the *measure of regular variation* associated with (\mathbf{Q}, \mathbf{a}) .

The regular variation of a random vector $\mathbf{Q} \in \mathbb{R}^d$ implies that its one-dimensional projections have regularly varying tails of a similar structure. More precisely, if \mathbf{Q} is regularly varying then for any $\mathbf{x} \in \mathbb{R}^d$,

$$\lim_{t \rightarrow \infty} \frac{P(\mathbf{Q} \cdot \mathbf{x} > t)}{t^{-\alpha} L(t)} = w(\mathbf{x}) \quad (4)$$

for a slowly varying function L and some $w(\mathbf{x}) : \mathbb{R}^d \rightarrow \mathbb{R}$ which is not identically zero. The property (4) was used as a definition of regular variation in [9], and it turns out to be equivalent to (3) for all non-integer α as well as for odd integers provided that \mathbf{Q} has non-negative components with a positive probability [31]. The question whether (4) and (3) are equivalent for even integers α in higher dimensions remains open.

In this paper we impose the following conditions on the coefficients $(\mathbf{Q}_n, M_n)_{n \in \mathbb{Z}}$.

Assumption 2. Let $(Z_n)_{n \in \mathbb{Z}}$ be an irreducible Markov chain with transition matrix H and stationary distribution π defined on a finite state space \mathcal{D} . Suppose that the coefficients $(\mathbf{Q}_n, M_n)_{n \in \mathbb{Z}}$ in (1) are induced by the stationary sequence $(Z_n)_{n \in \mathbb{Z}}$, Assumption 1 is satisfied, and, in addition, there exist a constant $\alpha > 0$ and a regularly varying function $\mathbf{a} : \mathbb{R} \rightarrow \mathbb{R}$ such that

(A3) For each $i \in \mathcal{D}$, $\mathbf{Q}_{0,i} \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ with an associated measure of regular variation μ_i .

(A4) $\Lambda(\beta) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log E(\|\Pi_{-n}\|^\beta) < 0$ for some $\beta > \alpha$. In particular,

$$\text{There exists } m > 0 \text{ such that } E(\|\Pi_{-m}\|^\alpha) < 1 \text{ and } E(\|\Pi_{-m}\|^\beta) < 1. \quad (5)$$

3. Main result

The following theorem extends results of [11, 12, 16, 23] to multivariate recursions of the form (1) with Markov-dependent coefficients.

Theorem 1. Let Assumptions 2 hold. Then $\mathbf{X} \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ with measure of regular variation $\mu_{\mathbf{X}}(\cdot) = \sum_{k=-\infty}^0 E(\mu_{z_k} \circ \Pi_{k+1}^{-1}(\cdot))$, where $\mu \circ \Pi^{-1}(\cdot)$ stands for $\mu(\{\mathbf{x} : \Pi \mathbf{x} \in \cdot\})$.

The theorem is an instance of the phenomenon “regular variation in, regular variation out” for the model (1). We remark that the mechanisms leading to regularly varying tails of \mathbf{X} are quite different in [12, 11] versus [9, 10]. In the former case, Kesten’s “critical exponent” is not available, and therefore more explicit assumptions about distribution of \mathbf{Q}_n are made. Then \mathbf{Q}_n dominates and creates cumulative effects, namely \mathbf{X} turns out to be regularly varying as a sum of regularly varying terms $\Pi_{n+1} \mathbf{Q}_n$. The setup of Assumption 2 is particularly appealing because a similar “cumulative effect” enables one to gain insight into the structure and fine properties of the sequence $(\mathbf{X}_n)_{n \in \mathbb{N}}$, in particular into the asymptotic behavior of both the partial sums as well as multivariate extremes of $(\mathbf{X}_n)_{n \in \mathbb{N}}$, see for instance [6, 32, 33, 34, 35, 36].

The proof of Theorem 1 is deferred to the Appendix which is included at the end of this paper. The proof combines ideas developed in [11], [16], and [23]. We notice that Grey conjectured in [11] that using his method it may be possible to extend the results of [16] and rid of the assumption that $(\mathbf{Q}_n)_{n \in \mathbb{Z}}$ and $(M_n)_{n \in \mathbb{Z}}$ are independent. We accomplish here the program suggested by Grey, and in fact extend it further to coefficients induced by a finite-state irreducible Markov chains.

4. Concluding remarks

The stochastic difference equation (1), in particular regular variation of the distribution tails of its stationary solutions, has been studied by many authors. The equation has a remarkable variety of both theoretical as well as real-world applications. For examples in theoretical probability see for instance [37, 38, 39, 40, 41]. For examples of applications in economics see for instance [18, 19, 20, 42, 43]. In this paper we showed that the tails of the stationary solution to the multivariate equation are regularly varying in a Markovian “regular variation in, regular variation out” setup, extending known i.i.d results. The main result of the paper is stated in Theorem 1. The extension to Markovian coefficients seems to be desirable in many, especially financial, applications.

Acknowledgements

We are very grateful to Krishna Athreya for the careful reading of a preliminary draft of this paper and many helpful remarks and suggestions. We would like to thank the anonymous Referee and the Associate Editor for helping us to significantly improve the presentation of this paper.

- [1] A. Brandt, P. Franken, B. Lisek, *Stationary Stochastic Models*, Wiley, Chichester, 1990.
- [2] R. F. Engle, *ARCH. Selected Readings*, Oxford Univ. Press, 1995.
- [3] T. Mikosch, C. Starica, Limit theory for the sample autocorrelations and extremes of a GARCH(1,1) process, *Ann. Statist.* 28 (2000) 1427–1451. An extended version is available at www.math.ku.dk/slash/~mikosch.
- [4] P. Diaconis, D. Freedman, Iterated random functions, *SIAM Rev.* 41 (1999) 45–76.
- [5] P. Embrechts, C. M. Goldie, Perpetuities and random equations, in: P. Mandl, M. Hušková (Eds.), *Asymptotic Statistics, 5th. Symp.* (Prague, 1993), *Contrib. Statist., Physica, Heidelberg*, 1994, pp. 75–86.
- [6] S. T. Rachev, G. Samorodnitsky, Limit laws for a stochastic process and random recursion arising in probabilistic modeling, *Adv. in Appl. Probab.* 27 (1995) 185–202.
- [7] W. Vervaat, On a stochastic difference equations and a representation of non-negative infinitely divisible random variables, *Adv. in Appl. Probab.* 11 (1979) 750–783.
- [8] A. Brandt, The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients, *Adv. in Appl. Probab.* 18 (1986) 211–220.
- [9] H. Kesten, Random difference equations and renewal theory for products of random matrices, *Acta. Math.* 131 (1973) 208–248.
- [10] C. M. Goldie, Implicit renewal theory and tails of solutions of random equations, *Ann. Appl. Probab.* 1 (1991) 126–166.
- [11] D. R. Grey, Regular variation in the tail of solutions of random difference equations, *Ann. Appl. Probab.* 4 (1994) 169–183.
- [12] A. K. Grincevičius, One limit distribution for a random walk on the line, *Lithuanian Math. J.* 15 (1975) 580–589.
- [13] B. de Saporta, Y. Guivarc’h, E. L. Page, On the multidimensional stochastic equation $Y_{n+1} = A_n Y_n + B_n$, *C. R. Math. Acad. Sci. Paris* 339 (2004) 499–502.
- [14] Y. Guivarc’h, Heavy tail properties of stationary solutions of multidimensional stochastic recursions, in: *Dynamics & Stochastics*, volume 48 of *IMS Lecture Notes Monogr.*, Inst. Math. Statist., Beachwood, OH, 2006, pp. 85–99.
- [15] C. Klüppelberg, S. Pergamenchtchikov, The tail of the stationary distribution of a random coefficient AR(q) model, *Ann. Appl. Probab.* 14 (2004) 971–1005.
- [16] S. I. Resnick, E. Willekens, Moving averages with random coefficients and random coefficient autoregressive models, *Comm. Statist. Stochastic Models* 7 (1991) 511–525.
- [17] R. Stelzer, Multivariate Markov-switching ARMA processes with regularly varying noise, *J. Multivariate Anal.* 99 (2008) 1177–1190.
- [18] J. Benhabib, A. Bisin, Z. Zhu, The distribution of wealth and fiscal policy in economies with finitely lived agents, 2010. The preprint is available at <http://www.econ.nyu.edu/user/benhabib/research.htm>.
- [19] S. Perrakis, C. Henin, The evaluation of risky investments with random timing of cash returns, *Management Sci.* 21 (1974) 79–86.
- [20] J. F. Collamore, Random recurrence equations and ruin in a Markov-dependent stochastic economic environment, *Ann. Appl. Probab.* 19 (2009) 1404–1458.
- [21] B. de Saporta, Tails of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ with Markovian coefficients, *Stochastic Process. Appl.* 115 (2005) 1954–1978.
- [22] A. Roitershtein, One-dimensional linear recursions with Markov-dependent coefficients, *Ann. Appl. Probab.* 17 (2007) 572–608.
- [23] A. P. Ghosh, D. Hay, V. Hirpara, R. Rastegar, A. Roitershtein, A. Schulteis, J. Suh, Random linear recursions with dependent coefficients, *Statist. Probab. Lett.* 80 (2010) 1597–1605.
- [24] M. Babilot, P. Bougerol, L. Elie, The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case, *Ann. Probab.* 25 (1997) 478–493.
- [25] Y. Ephraim, N. Merhav, Hidden Markov processes, *IEEE Trans. Inform. Theory* 48 (2002) 1518–1569.

- [26] S. I. Resnick, A. Subramanian, Heavy tailed hidden semi-Markov models, *Stoch. Models* 14 (1998) 319–334.
- [27] S. I. Resnick, *Extreme Values, Regular Variation and Point Processes*, Springer, New York, 1987.
- [28] S. I. Resnick, On the foundations of multivariate heavy tail analysis, *J. Appl. Probab.* 41 (2004) 191–212.
- [29] S. I. Resnick, Point processes, regular variation and weak convergence, *Adv. in Appl. Probab.* 18 (1986) 66–138.
- [30] F. Lindskog, Multivariate extremes and regular variation for stochastic processes, Ph.D. thesis, Zürich, Switzerland, 2004. Available from: www.e-collection.ethbib.ethz.ch/diss/.
- [31] B. Basrak, R. A. Davis, T. Mikosch, A characterization of multivariate regular variation, *Ann. Appl. Probab.* 12 (2000) 908–920.
- [32] A. A. Borovkov, K. A. Borovkov, *Asymptotic Analysis of Random Walks: Heavy-Tailed Distributions*, volume 118 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge UK, 2008.
- [33] R. A. Davis, T. Hsing, Point process and partial sum convergence for weakly dependent random variables with infinite variance, *Ann. Probab.* 23 (1995) 879–917.
- [34] L. de Haan, S. I. Resnick, H. Rootzén, C. G. de Vries, Extremal behavior of solutions to a stochastic difference equation with applications to ARCH processes, *Stochastic Process. Appl.* 32 (1989) 213–224.
- [35] D. G. Konstantinides, T. Mikosch, Large deviations for solutions to stochastic recurrence equations with heavy-tailed innovations, *Ann. Probab.* 33 (2005) 1992–2035.
- [36] R. Rastegar, V. Roytershteyn, A. Roitershtein, J. Suh, Discrete-time Langevin motion of a particle in a Gibbsian random potential, 2010. The preprint is available at <http://www.public.iastate.edu/~roiterst/papers/langevin4.pdf>.
- [37] D. R. Grey, L. Zhunwei, The asymptotic behaviour of extinction probability in the Smith-Wilkinson branching process, *Adv. in Appl. Probab.* 25 (1993) 263–289.
- [38] D. R. Grey, L. Zhunwei, The fractional linear probability generating function in the random environment branching process, *J. Appl. Probab.* 31 (1994) 38–47.
- [39] R. G. and U. Rösler, Asymptotic distribution theory for Hoare’s selection algorithm, *Adv. in Appl. Probab.* 28 (1996) 252–269.
- [40] E. S. Key, Recurrence and transience criteria for random walk in a random environment, *Ann. Probab.* 12 (1984) 529–560.
- [41] E. Mayer-Wolf, A. Roitershtein, O. Zeitouni, Limit theorems for one-dimensional transient random walks in Markov environments, *Ann. Inst. H. Poincaré-Probab. Statist.* 40 (2004) 635–659.
- [42] X. Gabaix, Zipf’s law for cities: an explanation, *Quarterly Journal of Economics* 114 (1999) 739–767.
- [43] U. Horst, Stability of linear stochastic difference equations in strategically controlled random environments, *Adv. in Appl. Probab.* 35 (2003) 961–981.
- [44] B. Basrak, R. A. Davis, T. Mikosch, Regular variation of GARCH processes, *Stochastic Process. Appl.* 99 (2002) 95–115.

Appendix. Proof of Theorem 1

The following result extends Lemma 2 in [11] and the relation (2.4) in [16]. Notice, that in contrast to [16] we do not assume that \mathbf{Q} and M are independent.

Lemma 1. *Let \mathbf{Y}, \mathbf{Q} be random d -vectors and Π be a random $d \times d$ matrix such that*

- (i) \mathbf{Q} is independent of the pair (\mathbf{Y}, Π)
- (ii) For some constant $\alpha > 0$ and regularly varying $\mathbf{a} : \mathbb{R} \rightarrow \mathbb{R}$, \mathbf{Y} and \mathbf{Q} belong to $\mathcal{R}_{d,\alpha,\mathbf{a}}$ with associated measures of regular variation measures ν and μ , respectively.
- (iii) $E(\|\Pi\|^\beta) < \infty$ for some $\beta > \alpha$.

Then, $\mathbf{Y} + \Pi\mathbf{Q} \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ with associated measure of regular variation $\nu(\cdot) + E(\mu \circ \Pi^{-1}(\cdot))$.

Proof. We need to show that for any compact set $K \subset S^{d-1}$,

$$\limsup_{n \rightarrow \infty} nP(\|\mathbf{Y} + \Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in K) \leq t^{-\alpha} [\mathfrak{S}_{\mathbf{Y}}(K) + E(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(K))] \quad (6)$$

while for any open set $G \subset S^{d-1}$,

$$\liminf_{n \rightarrow \infty} nP(\|\mathbf{Y} + \Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in G) \geq t^{-\alpha} [\mathfrak{S}_{\mathbf{Y}}(G) + E(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(G))] \quad (7)$$

To this end, we will use a decomposition resembling the one exploited in [11, Lemma 2] and [23, Proposition 2.1]. Namely, we fix $\varepsilon > 0$ and write for any Borel set $A \subset S^{d-1}$, $nP(\|\mathbf{Y} + \Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in A) = J_{t,A}^{(1)}(n) - J_{t,A}^{(2)}(n) + J_{t,A}^{(3)}(n) + J_{t,A}^{(4)}(n)$, where

$$\begin{aligned} J_{t,A}^{(1)}(n) &= nP(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in A), \\ J_{t,A}^{(2)}(n) &= nP(\|\mathbf{Y}\| > (1+\varepsilon)ta_n, \|\mathbf{Y} + \Pi\mathbf{Q}\| \leq ta_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in A) \\ J_{t,A}^{(3)}(n) &= nP((1-\varepsilon)ta_n < \|\mathbf{Y}\| \leq (1+\varepsilon)ta_n, \|\mathbf{Y} + \Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in A) \\ J_{t,A}^{(4)}(n) &= nP(\|\mathbf{Y}\| \leq (1-\varepsilon)ta_n, \|\mathbf{Y} + \Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in A). \end{aligned}$$

Fix a constant $\delta \in (0, 1)$ and let $K \subset S^{d-1}$ be an arbitrary compact set. Then $J_{t,K}^{(1)}(n) \leq nP(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y}} \in K^\delta) + nP(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y} + \Pi\mathbf{Q}}\| > \delta)$. It is not hard to check that for any constant $\gamma > 0$ and vectors $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{R}}_0^d$,

$$\|\overline{\mathbf{y}} - \overline{\mathbf{x} + \mathbf{y}}\| > \gamma \text{ implies } \|\mathbf{x}\| > \frac{\gamma\|\mathbf{y}\|}{2 + \gamma}. \quad (8)$$

Thus $nP(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y} + \Pi\mathbf{Q}}\| > \delta) \leq nP(\|\mathbf{Y}\| > ta_n, \|\Pi\|\|\mathbf{Q}\| > \frac{\delta ta_n}{3}) \leq nP(\|\mathbf{Y}\| > ta_n)P(\|\mathbf{Q}\| \geq \frac{\delta ta_n}{3}) + nP(\|\Pi\| \geq a_n^{\frac{\alpha+\beta}{2\beta}})$. Since $P(\|\Pi\| \geq a_n^{\frac{\alpha+\beta}{2\beta}}) \leq a_n^{-\frac{\alpha+\beta}{2}} E(\|\Pi\|^\beta)$, we have $\limsup_{n \rightarrow \infty} nP(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y} + \Pi\mathbf{Q}}\| > \delta) = 0$. Thus

$$\limsup_{n \rightarrow \infty} J_{t,K}^{(1)}(n) \leq \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} nP(\|\mathbf{Y}\| > t(1+\varepsilon)a_n, \overline{\mathbf{Y}} \in K^\delta) = t^{-\alpha} \mathfrak{S}_{\mathbf{Y}}(K). \quad (9)$$

Since $J_{t,K}^{(2)}(n) \leq nP(\|\Pi\| \geq ta_n^{\frac{\alpha+\beta}{2\beta}}) + nP(\|\mathbf{Y}\| > (1+\varepsilon)ta_n)P(\|\mathbf{Q}\| \geq \varepsilon ta_n^{\frac{\beta-\alpha}{2\beta}})$, we have

$$\limsup_{n \rightarrow \infty} J_{t,K}^{(2)}(n) = 0. \quad (10)$$

Next, $J_{t,K}^{(3)}(n) \leq nP((1-\varepsilon)ta_n < \|\mathbf{Y}\| \leq (1+\varepsilon)ta_n) \sim t^{-\alpha} [(1-\varepsilon)^{-\alpha} - (1+\varepsilon)^{-\alpha}]$. Hence

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} J_{t,K}^{(3)}(n) = 0. \quad (11)$$

Define $g_n(\mathbf{x}, A) = nP(\|\mathbf{Y} + \Pi\mathbf{Q}\| > ta_n, \overline{\mathbf{Y} + \Pi\mathbf{Q}} \in K | \mathbf{Y} = \mathbf{x}, \Pi = A)$. Fix constants $\rho > 0$ and $\eta > 0$, and let $J_{t,K}^{(4)}(n) = J_{t,K}^{(4,1)}(n) + J_{t,K}^{(4,2)}(n) + J_{t,K}^{(4,3)}(n)$, where

$$\begin{aligned} J_{t,K}^{(4,1)}(n) &= E(g(\mathbf{Y}, \Pi) \mathbf{I}_{\{\|\mathbf{Y}\| \leq (1-\varepsilon)ta_n\}} \mathbf{I}_{\{\|\Pi\| > \rho\}}) \\ J_{t,K}^{(4,2)}(n) &= E(g(\mathbf{Y}, \Pi) \mathbf{I}_{\{\|\mathbf{Y}\| \leq (1-\varepsilon)ta_n\}} \mathbf{I}_{\{\|\Pi\| \leq \rho\}} \mathbf{I}_{\{\|\mathbf{Y}\| > \eta\}}) \\ J_{t,K}^{(4,3)}(n) &= E(g(\mathbf{Y}, \Pi) \mathbf{I}_{\{\|\mathbf{Y}\| \leq (1-\varepsilon)ta_n\}} \mathbf{I}_{\{\|\Pi\| \leq \rho\}} \mathbf{I}_{\{\|\mathbf{Y}\| \leq \eta\}}). \end{aligned}$$

The first two terms tend to zero as η and ρ go to infinity. More precisely,

$$\begin{aligned} \limsup_{n \rightarrow \infty} J_{t,K}^{(4,1)}(n) &\leq \limsup_{n \rightarrow \infty} E\left(nP(\|\Pi\| \cdot \|\mathbf{Q}\| > \varepsilon ta_n | \Pi) \mathbf{I}_{\{\|\Pi\| > \rho\}}\right) \\ &= (\varepsilon t)^{-\alpha} E(\|\Pi\|^\alpha \mathbf{I}_{\{\|\Pi\| > \rho\}}) \rightarrow_{\rho \rightarrow \infty} 0, \end{aligned} \quad (12)$$

$$\begin{aligned} \limsup_{n \rightarrow \infty} J_{t,K}^{(4,2)}(n) &\leq \limsup_{n \rightarrow \infty} E(nP(\|\Pi\| \cdot \|\mathbf{Q}\| > \varepsilon ta_n | \Pi) \mathbf{I}_{\{\|\Pi\| \leq \rho\}} \mathbf{I}_{\{\|\mathbf{Y}\| > \eta\}}) \\ &\leq \rho^\alpha (\varepsilon t)^{-\alpha} P(\|\mathbf{Y}\| > \eta) \rightarrow_{\eta \rightarrow \infty} 0. \end{aligned} \quad (13)$$

To show the asymptotic of $J_{t,K}^{(4,3)}(n)$ as n goes to infinity write,

$$\begin{aligned} J_{t,K}^{(4,3)}(n) &\leq nP(\eta + \|\Pi\mathbf{Q}\| > ta_n, \overline{\Pi\mathbf{Q}} \in K^\delta) \\ &\quad + nP(\overline{\mathbf{Y} + \Pi\mathbf{Q}} - \overline{\Pi\mathbf{Q}} > \delta, \|\Pi\mathbf{Q}\| \geq \varepsilon ta_n, \|\mathbf{Y}\| \leq \eta). \end{aligned} \quad (14)$$

Applying the multivariate Breiman's lemma (see for instance [44, Proposition 5.1]) to the first term in the right-hand side of the last inequality and (8) to the second, we obtain $\limsup_{n \rightarrow \infty} J_{t,K}^{(4,3)}(n) \leq t^{-\alpha} E(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(K))$. Thus (6) is implied by (9)-(14).

It remains to show that (7) holds for any open set $G \subset S^{d-1}$. According to (10), $\limsup_{n \rightarrow \infty} J_{t,G}^{(2)}(n) \leq \limsup_{n \rightarrow \infty} J_{t,G}^{(2)}(n) = 0$. Let $G_k \subset S^{d-1}$, $k \in \mathbb{N}$ be open sets such that $G_k \subset \overline{G_k} \subset G_{k+1} \subset G$. Let $\gamma_k = \frac{1}{2} \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x} \in G_k, \mathbf{y} \in G^c\}$. Then, $J_{t,G}^{(1)}(n) \geq nP(\|\mathbf{Y}\| > t(1 + \varepsilon)a_n, \overline{\mathbf{Y}} \in G_k) - nP(\|\mathbf{Y}\| > t(1 + \varepsilon)a_n, \|\overline{\mathbf{Y}} - \overline{\mathbf{Y} + \Pi\mathbf{Q}}\| > \gamma_k)$. By (8), $\liminf_{n \rightarrow \infty} J_{t,G}^{(1)}(n) \geq \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} nP(\|\mathbf{Y}\| > t(1 + \varepsilon)a_n, \overline{\mathbf{Y}} \in G_k) = t^{-\alpha} \mathfrak{S}_{\mathbf{Y}}(G_k)$. Letting $k \rightarrow \infty$ we obtain $\liminf_{n \rightarrow \infty} J_{t,G}^{(1)}(n) \geq t^{-\alpha} \mathfrak{S}_{\mathbf{Y}}(G)$. To conclude, observe that

$$\begin{aligned} J_{t,G}^{(4,3)}(n) &\geq nP(\|\Pi\mathbf{Q}\| - \eta > ta_n, \overline{\Pi\mathbf{Q}} \in G_k) - nP(\rho\|\mathbf{Q}\| - \eta > ta_n; \|\mathbf{Q}\| \leq \eta) \\ &\quad - nP(\|\overline{\mathbf{Y} + \Pi\mathbf{Q}} - \overline{\Pi\mathbf{Q}}\| > \gamma_k, \|\Pi\mathbf{Q}\| \geq \varepsilon ta_n, \|\mathbf{Y}\| \leq \eta). \end{aligned}$$

By (8), $\liminf_{n \rightarrow \infty} J_{t,G}^{(4,3)}(n) \geq t^{-\alpha} E(\mathfrak{S}_{\mathbf{Q}} \circ \Pi^{-1}(G_k))$. Letting $k \rightarrow \infty$ establishes (7). \square

The next lemma, which generalizes Proposition 2.1 of [23], is the key element of our proof of Theorem 1.

Lemma 2. *Let Assumption 2 hold. Fix an integer $k \leq -1$ and let $\mathbf{Y}_{k+1} \in \mathbb{R}^d$ be a random vector such that $\mathbf{Y}_{k+1} \in \sigma(Z_n, \mathbf{Q}_n, M_n : n \geq k+1)$. Let $\mathbf{Y}_k = \mathbf{Y}_{k+1} + \Pi_{k+1} \mathbf{Q}_k$ and write $\mathbf{Y}_k = \sum_{i \in \mathcal{D}} \mathbf{Y}_{k,i} \mathbf{I}_{\{Z_k=i\}}$.*

Then, each vector $\mathbf{Y}_{k,i}$ belongs to $\mathcal{R}_{d,\alpha,\mathbf{a}}$ with associated measure of regular variation $\nu_{k,i} := E(\nu_{k+1,Z_{k+1}}(\cdot) \mathbf{I}_{\{Z_k=i\}}) + E(\mu_i \circ \Pi_{k+1}^{-1}(\cdot) \mathbf{I}_{\{Z_k=i\}})$, and hence $\mathbf{Y}_k \in \mathcal{R}_{d,\alpha,\mathbf{a}}$ with associated measure of regular variation $E(\nu_{k+1,Z_{k+1}}(\cdot)) + E(\mu_{Z_k} \circ \Pi_{k+1}^{-1}(\cdot))$.

Proof. Since $P((\mathbf{Q}_k, \Pi_{k+1}, \mathbf{Y}_{k+1}) \in \cdot | Z_{k+1} = i, Z_k = j) = P((\mathbf{Q}_{1,j}, \Pi_{k+1,i}, \mathbf{Y}_{k+1,i}) \in \cdot)$,

using Lemma 1 we obtain for Borel subsets $A \subset \overline{\mathbb{R}}_0^d$,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} P(a_n^{-1} \mathbf{Y}_{k,i} \in A) \\
&= \lim_{n \rightarrow \infty} \sum_{j \in \mathcal{D}} P(\mathbf{Y}_{k+1} + \Pi_{k+1} \mathbf{Q}_k \in a_n A \mid Z_{k+1} = j, Z_k = i) \pi_i H(i, j) \\
&= \lim_{n \rightarrow \infty} \sum_{j \in \mathcal{D}} P(\mathbf{Y}_{k+1,j} + \Pi_{k+1,j} \mathbf{Q}_{k,i} \in a_n A) \pi_i H(i, j). \\
&= \sum_{j \in \mathcal{D}} [\nu_{k+1,j}(A) + E(\mu_i \circ \Pi_{k+1,j}^{-1}(A))] \pi_i H(i, j) \\
&= E(\nu_{k+1,Z_{k+1}}(A) \mathbf{I}_{\{Z_k=i\}}) + E(\mu_i \circ \Pi_{k+1}^{-1}(A) \mathbf{I}_{\{Z_k=i\}}).
\end{aligned}$$

The proof of the lemma is completed. \square

We are now in position to complete the proof of Theorem 1. First we introduce some notations. Throughout the rest of the paper:

For a constant $\delta > 0$ and a set K (either in S^{d-1} or $\overline{\mathbb{R}}_0^d$), let K^δ denote the closed δ -neighborhood of K , that is $K^\delta = \{\mathbf{x} : \exists \mathbf{y} \in K \text{ s.t. } \|\mathbf{x} - \mathbf{y}\| \leq \delta\}$. For $\mathbf{x} \in \mathbb{R}^d / \{0\}$, let $\bar{\mathbf{x}}$ denote its direction $\mathbf{x} / \|\mathbf{x}\|$. For a set G , let \bar{G} denote its closure $\bigcap_{\delta > 0} G^\delta$.

The final step in the proof is similar to the corresponding argument in [16], and is reproduced here for the sake of completeness. It follows from Lemma 2 that, for any $L \in \mathbb{N}$ and Borel $A \subset \overline{\mathbb{R}}_0^d$,

$$\lim_{n \rightarrow \infty} nP\left(\sum_{k=-L}^0 \Pi_{k+1} \mathbf{Q}_k \in a_n A\right) = \sum_{k=-L}^0 E(\mu_{z_k} \circ \Pi_{k+1}^{-1}(A)), \quad (15)$$

while [23, Theorem 1.4] yields with the help of (5) that for any constant $\delta > 0$,

$$\lim_{L \rightarrow \infty} \limsup_{n \rightarrow \infty} nP\left(\sum_{k=-\infty}^{-L-1} \|\Pi_{k+1}\| \cdot \|\mathbf{Q}_k\| > \delta a_n\right) = 0. \quad (16)$$

For a compact set $K \subset \overline{\mathbb{R}}_0^d$, we have

$$P\left(\sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in a_n K\right) \leq P\left(\sum_{k=-L}^0 \mathbf{Q}_k \Pi_{k+1} \in a_n K^\delta\right) + P\left(\sum_{k=-\infty}^{-L-1} \|\Pi_{k+1} \mathbf{Q}_k\| > \delta a_n\right).$$

Hence, $\limsup_{n \rightarrow \infty} nP\left(\frac{1}{a_n} \sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in K\right) \leq \sum_{k=-L}^0 E(\mu_{z_k} \circ \Pi_{k+1}^{-1}(K^\delta))$ in virtue of (15) and (16). Letting then $\delta \rightarrow 0$, we obtain

$$\limsup_{n \rightarrow \infty} nP\left(a_n^{-1} \sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in K\right) \leq \sum_{k=-\infty}^0 E(\mu_{z_k} \circ \Pi_{k+1}^{-1}(K)). \quad (17)$$

Let $G \subset \overline{\mathbb{R}}_0^d$ be relatively compact and open. Consider open relatively compact sets $G_k \subset \overline{\mathbb{R}}_0^d$, $k \in \mathbb{N}$, such that $G_k \subset \bar{G}_k \subset G_{k+1} \subset G$. For any m, L , there is $\varepsilon > 0$ such that

$$\left\{ \sum_{k=-L}^0 \Pi_{k+1} \mathbf{Q}_k \in a_n G_m \right\} \cup \left\{ \left\| \sum_{k=-\infty}^{-L-1} \Pi_{k+1} \mathbf{Q}_k \right\| \leq \varepsilon a_n \right\} \subset \left\{ \sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in a_n G \right\}.$$

Therefore, with $\mathcal{F}_0 := \sigma(M_n, Z_n : n \leq 0)$, we have for any G_m ,

$$\begin{aligned} \liminf_{n \rightarrow \infty} nP \left(a_n^{-1} \sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in G \right) &= \liminf_{n \rightarrow \infty} nE \left[P \left(a_n^{-1} \sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in G \mid \mathcal{F}_0 \right) \right] \\ &= \liminf_{n \rightarrow \infty} E \left[nP \left(a_n^{-1} \sum_{k=-L}^0 \Pi_{k+1} \mathbf{Q}_k \in G_m \mid \mathcal{F}_0 \right) P \left(\left\| \sum_{k=-\infty}^{-L-1} \Pi_{k+1} \mathbf{Q}_k \right\| \leq \varepsilon a_n \mid \mathcal{F}_0 \right) \right] \\ &\geq E \left[\liminf_{n \rightarrow \infty} nP \left(a_n^{-1} \sum_{k=-L}^0 \Pi_{k+1} \mathbf{Q}_k \in G_m \mid \mathcal{F}_0 \right) P \left(\left\| \sum_{k=-\infty}^{-L-1} \Pi_{k+1} \mathbf{Q}_k \right\| \leq a_n \varepsilon \mid \mathcal{F}_0 \right) \right], \end{aligned}$$

where for the last inequality we used Fatou's lemma. Hence, (15) yields the lower bound $\liminf_{n \rightarrow \infty} nP \left(a_n^{-1} \sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in G \right) \geq \sum_{k=-L}^0 E(\mu_{z_k} \circ \Pi_{k+1}^{-1}(G_m))$. Letting $m \rightarrow \infty$ and then $L \rightarrow \infty$, $\liminf_{n \rightarrow \infty} nP \left(a_n^{-1} \sum_{k=-\infty}^0 \Pi_{k+1} \mathbf{Q}_k \in G \right) \geq \sum_{k=-\infty}^0 E(\mu_{z_k} \circ \Pi_{k+1}^{-1}(G))$. This bound along with (17) yield the claim of the theorem provided that we have shown that $\mu_{\mathbf{X}}(\cdot) = E(\mu_{z_k} \circ \Pi_{k+1}^{-1}(\cdot))$ is a Radon measure on $\overline{\mathbb{R}}_0^d$, that is (see for instance Remark 3.3 in [17]) $\mu_{\mathbf{X}}(K) < \infty$ for any compact set $K \in \overline{\mathbb{R}}_0^d$. Toward this end notice that since $\epsilon_K := \inf_{\mathbf{x} \in K} \|\mathbf{x}\| > 0$ and in virtue of (A4) of Assumption 2,

$$\begin{aligned} \mu_{\mathbf{X}}(K) &\leq \sum_{k=-\infty}^0 E \left[\sum_{i \in \mathcal{D}} \mu_i \circ \Pi_{k+1}^{-1}(K) \right] = \sum_{k=-\infty}^0 E \left[\sum_{i \in \mathcal{D}} \mu_i(\{\mathbf{x} : \Pi_{k+1} \mathbf{x} \in K\}) \right] \\ &\leq \sum_{k=-\infty}^0 E \left[\sum_{i \in \mathcal{D}} \mu_i(\{\mathbf{x} : \|\mathbf{x}\| \geq \epsilon_K \|\Pi_{k+1}\|^{-1}\}) \right] = \sum_{k=-\infty}^0 |\mathcal{D}| \epsilon_K^{-\alpha} E(\|\Pi_{k+1}\|^\alpha) < \infty, \end{aligned}$$

completing the proof of the theorem. \square