Random linear recursions with dependent coefficients[☆]

Arka P. Ghosh[1,2], Diana Hay[1], Vivek Hirpara[1,3], Reza Rastegar[1], Alexander Roitershtein[1,*], Ashley Schulteis[4], Jiyeon Suh[5]

Abstract

We consider the equation $R_n = Q_n + M_n R_{n-1}$, with random non-i.i.d. coefficients $(Q_n, M_n)_{n \in \mathbb{Z}} \in \mathbb{R}^2$, and show that the distribution tails of the stationary solution to this equation are regularly varying at infinity.

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1. Introduction and statement of results

1.1. Outline

Let $(Q_n, M_n)_{n \in \mathbb{Z}}$ be $\mathbb{R}^2$-valued random pairs and consider the recursion

$$R_n = Q_n + M_n R_{n-1}, \qquad n \in \mathbb{N}, \; R_n \in \mathbb{R}. \tag{1}$$

This equation has a wide variety of real world and theoretical applications, see (Embrechts and Goldie, 1994; Vervaat, 1979). Sufficient (in fact, close to necessary) conditions for R_n to converge in law, independently of R_0, to $R = Q_0 + \sum_{n=1}^{\infty} Q_{-n} \prod_{i=0}^{n-1} M_{-i}$, can be found in (Brandt, 1986).

The distribution tails of R were shown to be regularly varying in (Kesten, 1973; Goldie, 1991; Grincevičius, 1975; Grey, 1994) provided that the pairs $(Q_n, M_n)_{n \in \mathbb{Z}}$ form an i.i.d. sequence. In the setup of (Kesten, 1973; Goldie, 1991) the tails are in fact power tailed. Recall that $f : \mathbb{R} \to \mathbb{R}$ is called regularly varying if $f(t) = t^\alpha L(t)$ for some $\alpha \in \mathbb{R}$ where $L(t)$ is a slowly varying function, that is $L(\lambda t) \sim L(t)$ for all $\lambda > 0$. Here and henceforth $f(t) \sim g(t)$ (as a rule, we omit "as $t \to \infty$") means $\lim_{t \to \infty} f(t)/g(t) = 1$.

[*]Corresponding author. E-mail: roiterst@iastate.edu
[1]Department of Mathematics, Iowa State University, Ames, IA 50011, USA
[2]Department of Statistics, Iowa State University, Ames, IA 50011, USA
[3]Department of Economics, Iowa State University, Ames, IA 50011, USA
[4]Department of Math, Physics and CS, Wartburg College, Waverly, IA 50677, USA
[5]Department of Math, Grand Valley State University, Allendale, MI 49401, USA

The mechanisms leading to regularly varying tails of R are different in (Kesten, 1973; Goldie, 1991) versus (Grincevičius, 1975; Grey, 1994). In the former case, there exists a critical exponent α such that $E(|M_n|^\alpha) = 1$ and $E(|Q_n|^\alpha) < \infty$. Then, R is heavy tailed essentially because of one atypical fluctuation of $S_n = \sum_{i=0}^{n-1} \log |M_{-i}|$, which follows from renewal arguments for S_n under exponential tilt. Remarkably, while no particular tail behavior is assumed on inputs, exact power law tails appear in the output. In contrast, the critical exponent is not available in the latter case, where the tails of Q_n are assumed to be regular varied. The second case thus provides an instance of the phenomenon "regular variation in, regular variation out" for (1). This setup is appealing because it enables one to gain insight into the structure and fine properties of the sequence $(R_n)_{n\in\mathbb{N}}$, including the asymptotic behavior of its partial sums and extremes, see for instance (de Haan et al., 1989; Davis and Hsing, 1995; Konstantinides and Mikosch, 2005; Rachev and Samorodnitsky, 1995; Rastegar et al., 2010).

The goal of this paper is to study (1) with non-i.i.d. coefficients. The extension is desirable in many, especially financial, applications. See (Perrakis and Henin, 1974; Collamore, 2009). We remark that an extension of the main result of (Kesten, 1973; Goldie, 1991) to a Markov setup has been obtained in (de Saporta, 2005; Roitershtein, 2007; Collamore, 2009).

Definition 1.1. *The coefficients* $(Q_n, M_n)_{n\in\mathbb{Z}}$ *are said to be induced by a sequence of random variables* $(X_n)_{n\in\mathbb{Z}}$, *each valued in a countable set* $\mathcal{D}$, *if there exist independent random variables* $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z},i\in\mathcal{D}} \in \mathbb{R}^2$ *such that for a fixed* $i \in \mathcal{D}$, $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z}}$ *are i.i.d,*

$$Q_n = \sum_{j\in\mathcal{D}} Q_{n,j} \mathbf{I}_{\{X_n=j\}} = Q_{n,X_n}, \quad M_n = \sum_{j\in\mathcal{D}} M_{n,j} \mathbf{I}_{\{X_n=j\}} = M_{n,X_n}, \tag{2}$$

and $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z},i\in\mathcal{D}}$ *is independent of* $(X_n)_{n\in\mathbb{Z}}$.

Note that if $(X_n)_{n\in\mathbb{Z}}$ is a finite Markov chain, then (2) defines a *Hidden Markov Model* (HMM), see (Ephraim and Merhav, 2002) for a survey of HMM and their applications. Heavy tailed HMM are considered for instance in (Resnick and Subramanian, 1998), see also references therein.

1.2. Markov-dependent coefficients: regular variation in, regular variation out

First, we will impose the following conditions on the coefficients in (1).

Assumption 1.2. *Assume that the sequence* $(Q_{n,i}, M_{n,i})_{n\in\mathbb{Z},i\in\mathcal{D}} \in \mathbb{R}^2$ *is induced by a stationary irreducible Markov chain* $(X_n)_{n\in\mathbb{Z}}$ *on a countable state space* $\mathcal{D}$. *Furthermore, suppose that there exists a constant* $\alpha > 0$ *s. t.:*

(A1) There is a slowly varying function $L(t)$ *and two sequences of constants* $(q_i^{(\eta)})_{i\in\mathcal{D}}$, $\eta \in \{-1, 1\}$, *such that* $\lim_{t\to\infty} \frac{P(Q_{n,i}\cdot\eta > t)}{t^{-\alpha}L(t)} = q_i^{(\eta)}$ *uniformly in* $i \in \mathcal{D}$. *Moreover, we have* $\sum_{j\in\mathcal{D}} q_j^{(1)} > 0$ *and* $\sup_{j\in\mathcal{D}} \max\{q_j^{(1)}, q_j^{(-1)}\} < \infty$.

(A2) There exists $\beta > \alpha$ *such that* $\sup_{i\in\mathcal{D}} E(|M_{0,i}|^\beta) < \infty$.

(A3) Let $m_i^{(\eta)} = E(|M_{0,i}|^\alpha \mathbf{I}_{\{M_{0,i}\cdot\eta > 0\}})$, $m_i = m_i^{(-1)} + m_i^{(1)}$. *Then* $\sup_{i\in\mathcal{D}} m_i < 1$.

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(A4) $\lim_{\varepsilon \to 0+} P(M_{1,i} \le \varepsilon) = P(M_{1,i} \le 0)$ *uniformly in* $i \in \mathcal{D}$.

Let $B_b = \left\{ \mathbf{x} = (x_n)_{n \in \mathcal{D}} : \sup_{n \in \mathcal{D}} |x_n| < +\infty \right\} \subset \mathbb{R}^{\mathcal{D}}$ be equipped with the norm $\|\mathbf{x}\| = \sup_{n \in \mathcal{D}} |x_n|$. Let H be transition matrix of the stationary backward chain X_{-n}, that is $H(i, j) = P(X_n = j | X_{n+1} = i)$. Define matrices G_η, $\eta \in \{-1, 1\}$, as $G_\eta(i, j) = m_i^{(\eta)} H(i, j)$ and set $G(i, j) = m_i H(i, j)$, $i, j \in \mathcal{D}$. The spectral radius of G (operator in B_b) is less than 1 by (A3).

Under Assumption 1.2, the tails of R and Q_0 have similar structure. Denote $\mathbf{q}^{(\eta)} = \left(q_i^{(\eta)} \right)_{i \in \mathcal{D}}$, $\mathbf{m}^{(\eta)} = \left(m_i^{(\eta)} \right)_{i \in \mathcal{D}}$, $\eta \in \{-1, 1\}$, and $\mathbf{q} = \mathbf{q}^{(1)} + \mathbf{q}^{(-1)}$.

Theorem 1.3. *Let Assumptions 1.2 hold and suppose that $P(M_{0,i} > 0) = 1$ for all $i \in \mathcal{D}$. Then, for all $i \in \mathcal{D}$, $P(R > t | X_0 = i) \sim K_i t^{-\alpha} L(t)$, where $\mathbf{K} = (K_i)_{i \in \mathcal{D}} \in B_b$ is defined by $\mathbf{K} = (I - G)^{-1} \mathbf{q}^{(1)}$.*

Theorem 1.3 yields its analog without the restriction $P(M_0 > 0) = 1$.

Theorem 1.4. *Let Assumption 1.2 hold. Then, $P(R \cdot \eta > t | X_0 = i) \sim K_i^{(\eta)} t^{-\alpha} L(t)$ for $\eta \in \{-1, 1\}$, where $\mathbf{K}^{(\eta)} = \left(K_i^{(\eta)} \right)_{i \in \mathcal{D}} \in B_b$ are given by*

$$\mathbf{K}^{(\eta)} = \frac{1}{2} \left((I - G)^{-1} \left(\mathbf{q}^{(1)} + \mathbf{q}^{(-1)} \right) + \eta (I - G_+ + G_-)^{-1} \left(\mathbf{q}^{(1)} - \mathbf{q}^{(-1)} \right) \right).$$

Theorem 1.3 and 1.4 are proved in Sections 2 and 3, respectively.

1.3. Kesten's power law for coefficients induced by chains of infinite order

We next consider coefficients induced by process with infinite memory.

Definition 1.5. *A C-chain is a stationary process $(X_n)_{n \in \mathbb{Z}}$ taking values in a finite set (alphabet) $\mathcal{D}$ such that*

(i) For any $i_1, i_2, \ldots, i_n \in \mathcal{D}$, $P(X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n) > 0$.

(ii) For any $i_0 \in \mathcal{D}$ and any sequence $(i_n)_{n \ge 1} \in \mathcal{D}^{\mathbb{N}}$, the following limit exists:

$$\lim_{n \to \infty} P(X_0 = i_0 | X_{-k} = i_k, \ 1 \le k \le n) = P(X_0 = i_0 | X_{-k} = i_k, \ k \ge 1),$$

where the right-hand side is a regular version of the conditional probabilities.

(iii) (fading memory) For $n \ge 0$ let

$$\gamma_n = \sup \left\{ \left| \frac{P(X_0 = i_0 | X_{-k} = i_k, \ k \ge 1)}{P(X_0 = j_0 | X_{-k} = j_k, \ k \ge 1)} - 1 \right| : i_k = j_k, \ k = 1, \ldots, n \right\}.$$

Then, the numbers γ_n are all finite and $\limsup_n \log \gamma_n / n < 0$.

C-chains are a particular case of *chains of infinite order* (*chains with complete connections*), see e.g. (Kaijser, 1981; Iosifesku and Grigoresku, 1990). The distributions of C-chains (a particular case of *g*-measures introduced in (Keane, 1972)) are Gibbs states in the sense of Bowen (also known as Dobrushin-Lanford-Ruelle states), see e.g. (Bowen, 1975; Lalley, 1986).

We have, see also (Berbee, 1987; Fernández and Maillard, 2004),

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Theorem A. *(Lalley, 1986) Let $(X_n)_{n\in\mathbb{Z}}$ be a C-chain with alphabet $\mathcal{D}$, $\mathcal{S} = \bigcup_{n\geq 1}\mathcal{D}^n$, and $\zeta : \mathcal{S} \to \mathcal{D}$ be the projection into the last coordinate, i.e. $\zeta\big((s_1,\ldots,s_n)\big) = s_n$. Then (in an enlarged probability space) there exist a stationary irreducible Markov chain $(Y_n)_{n\in\mathbb{Z}}$ on $\mathcal{S}$, and constants $r \in \mathbb{N}$, $\delta > 0$, s. t. the following Markov representation holds: $X_n = \zeta(Y_n)$, $n \in \mathbb{Z}$.*

Furthermore, for any $(y_n)_{n\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ and $s \in \mathbb{N}$,

(i) $P\big(Y_{n+1} = (x_1, x_2,\ldots,x_t)|Y_n = (y_1, y_2,\ldots,y_s)\big) = 0$ unless either $t = 1$ or $t = s+1$ and $x_i = y_i$ for all $i \leq s$,

(ii) $P\big(Y_{n+1} = (y)|Y_{n+1} \in \mathcal{D},\, Y_n = (y_1, y_2,\ldots,y_s)\big) = P\big(Y_0 = (y)|Y_0 \in \mathcal{D}\big).$

(iii) $P\big(Y_{n+1} \in \mathcal{D}|Y_n = (y_1, y_2,\ldots,y_{mr})\big) = \delta$, for all $m \in \mathbb{N}$.

Moreover, without loss of generality, we can assume that

*(iv) r is **even** (see the beginning of the proof in (Lalley, 1986, p. 1266)).*

The following result is proved in Section 4.

Theorem 1.6. *Assume that the coefficients $(Q_n, M_n)_{n\in\mathbb{Z}}$ are induced by a C-chain $(X_n)_{n\in\mathbb{Z}}$. Furthermore, suppose that*

(i) $|Q_0| < q_0$ and $m_0^{-1} < |M_0| < m_0$, a.s., for some constants $q_0 > 0$, $m_0 > 0$.

(ii) $\Lambda(\beta) := \limsup\limits_{n\to\infty} \frac{1}{n}\log E\left(\prod_{i=0}^{n-1}|M_i|^{\beta}\right) \geq 0$ changes its sign in $(0,\infty)$.

(iii) $P(\log|M_0| = \delta \cdot k \text{ for some } k \in \mathbb{Z}|M_0 \neq 0) < 1$ for all $\delta > 0$.
  Let $\mathcal{X} = (X_n)_{n\geq 0}$. Then, with $\alpha > 0$ defined below in (4):

(a) $\lim\limits_{t\to\infty} t^{\alpha}P(R\cdot\eta > t|\mathcal{X}) = K_\eta(\mathcal{X})$ for some bounded $K_\eta : \mathcal{D}^{\mathbb{Z}_+} \to \mathbb{R}_+$, $\eta \in \{-1,1\}$. Moreover, the convergence is uniform on $\mathcal{X}$.

(b) If $P(M_0 < 0) > 0$ then $P\big(K_1(\mathcal{X}) = K_{-1}(\mathcal{X})\big) = 1$.

(c) If $P(Q_0 > 0, M_0 > 0) = 1$ then $P\big(K_1(\mathcal{X}) > 0\big) = 1$.

(d) For $\eta \in \{-1,1\}$, if $P\big(K_\eta(\mathcal{X}) > 0\big) > 0$ then $P\big(K_\eta(\mathcal{X}) > 0\big) = 1$.

(e) $P\big(K_1(\mathcal{X}) = K_{-1}(\mathcal{X}) = 0\big) = 1$ iff there is a function $\Gamma : \mathcal{D} \to \mathbb{R}$ s. t.

$$P\big(Q_0 + \Gamma(X_0)M_0 = \Gamma(X_1)\big) = 1. \tag{3}$$

Moreover, if $P\big(Q_{0,i} = q, M_{0,i} = m\big) < 1$ for all pairs $(q, m) \in \mathbb{R}^2$ and any $i \in \mathcal{D}$, then $P\big(K_1(\mathcal{X}) = K_{-1}(\mathcal{X}) = 0\big) = 1$ iff $P\big(Q_0 + cM_0 = c\big) = 1$ for some $c \in \mathbb{R}$.

It turns out (see below, in Section 4) that the lim sup in (ii) is in fact a limit, and the parameter $\alpha > 0$ is (uniquely) determined by

$$\Lambda(\alpha) = 0, \quad \text{where } \Lambda(\beta) = \lim_{n\to\infty} \frac{1}{n}\log E\left(\prod_{i=0}^{n-1}|M_i|^{\beta}\right). \tag{4}$$

We remark that condition (3) is a natural generalization of the criterion that appears in the i.i.d. case, see Theorem B in Section 4 below.

Following the idea of (Mayer-Wolf et al., 2004), Theorem 1.6 is obtained in Section 4 rather directly from its i.i.d. prototype (cited as Theorem B below) by applying a

Doeblin's "cyclic trick" and associating R to a linear recursion with i.i.d. coefficients. Though a reduction to the main results of (Roitershtein, 2007) is possible, it would anyway entail considerable extra work. The approach taken here enjoys the finite range and mixing properties of X_n, in addition to the existence of Y_n. It is much "lighter" than those used in (Roitershtein, 2007) for general Markov chains and in (de Saporta, 2005) for a finite state case, both built on techniques of (Goldie, 1991).

We conclude with the remark that, using the Markov representation for C-chains, it is straightforward to deduce the following from Theorem 1.4.

Theorem 1.7. *Assume that the coefficients $(Q_n, M_n)_{n \in \mathbb{Z}}$ are induced by a C-chain $(X_n)_{n \in \mathbb{Z}}$. Furthermore, suppose that there exists $\alpha > 0$ such that:*

(i) There is a slowly varying function $L(t)$ and two sequences of constants $(q_i^{(\eta)})_{i \in \mathcal{D}}$, $\eta \in \{-1, 1\}$, such that for all $n \in \mathbb{Z}$ $\lim_{t \to \infty} \frac{P(Q_{n,i} \cdot \eta > t)}{t^{-\alpha} L(t)} = q_i^{(\eta)}$.

(ii) There exists $\beta > \alpha$ such that $\sup_{i \in \mathcal{D}} E(|M_{0,i}|^\beta) < \infty$.
Let $\mathcal{X} = (X_n)_{n \geq 0}$. Then, for some bounded function $K^{(\eta)}$, $\eta \in \{-1, 1\}$, we have $P(R \cdot \eta > t | \mathcal{X}) \sim K^{(\eta)}(\mathcal{X}) t^{-\alpha} L(t)$, a.s. Moreover, $P(K^{(1)}(\mathcal{X}) \neq 0) > 0$.

2. Proof of Theorem 1.3

The key to the result is Proposition 2.1 extending the corresponding statement in (Grincevičius, 1975; Grey, 1994).

Proposition 2.1. *Let Y be a random variable such that:*

(i) $Y \in \sigma(X_n, Q_{n,i}, M_{n,i} : n \leq 0, i \in \mathcal{D})$.

(ii) For $\eta \in \{-1, 1\}$ there exist non-negative constants $(c_i^{(\eta)})_{i \in \mathcal{D}}$ such that

    (a) $\lim_{t \to \infty} \frac{P(Y \cdot \eta > t | X_0 = i)}{t^{-\alpha} L(t)} = c_i^{(\eta)}$, uniform on $i \in \mathcal{D}$.

    (b) $\sup_{i \in \mathcal{D}} c_i^{(\eta)} < \infty$.

Then $\lim_{t \to \infty} \frac{P(Q_1 + M_1 Y > t | X_1 = i)}{t^{-\alpha} L(t)} = q_i^{(1)} + m_i^{(1)} \sum_{j \in \mathcal{D}} H(i,j) c_j^{(1)}$ uniformly on $i \in \mathcal{D}$.

Proof. Let $(Y_i)_{i \in \mathcal{D}}$ be random variables independent of both $(X_n)_{n \in \mathbb{Z}}$ and $(Q_{1,i}, M_{1,i})_{i \in \mathcal{D}}$, such that $P(Y_i \leq t) = P(Y \leq t | X_0 = i)$ for all $t \in \mathbb{R}$ and $i \in \mathcal{D}$. It follows from $P((Q_1, M_1, Y) \in \cdot | X_1 = i, X_0 = j) = P((Q_{1,i}, M_{1,i}, Y_j) \in \cdot)$ that

$$P(Q_1 + M_1 Y > t | X_1 = i) =$$
$$= \sum_{j \in \mathcal{D}} P(Q_1 + M_1 Y > t | X_1 = i, X_0 = j) H(i,j)$$
$$= \sum_{j \in \mathcal{D}} P(Q_{1,i} + M_{1,i} Y_j > t) H(i,j). \tag{5}$$

By (Grey, 1994, Lemma 2), which is the i.i.d. prototype of our proposition,

$$P(Q_{1,i} + M_{1,i} Y_j > t) \sim t^{-\alpha} L(t)\left(q_i^{(1)} + c_j^{(1)} m_i^{(1)}\right). \tag{6}$$

To complete the proof, it suffices to show that the convergence in (6) is uniform on i, j. To this end we decompose $P(Q_{1,i} + M_{1,i}Y_j > t)$ into individually tractable terms, as in (Grey, 1994, Lemma 2). Fix $\varepsilon \in (0, 1)$ and write

$$P(Q_{1,i} + M_{1,i}Y_j > t) = A_{i,j}^{(1)}(t) - A_{i,j}^{(2)}(t) + A_{i,j}^{(3)}(t) + A_{i,j}^{(4)}(t), \text{ where}$$

$A_{i,j}^{(1)}(t) = P(Q_{1,i} > t(1+\varepsilon))$, $A_{i,j}^{(2)}(t) = P(Q_{1,i} > (1+\varepsilon)t, Q_{1,i} + M_{1,i}Y_j \le t)$,
$A_{i,j}^{(3)}(t) = P(|Q_{1,i} - t| \le \varepsilon t, Q_{1,i} + M_{1,i}Y_j > t)$,
$A_{i,j}^{(4)}(t) = P(Q_{1,i} \le (1-\varepsilon)t, Q_{1,i} + M_{1,i}Y_j > t)$.
By (A1), $\frac{A_{i,j}^{(1)}(t)}{t^{-\alpha}L(t)}$ converges uniformly in i, j to $q_i^{(1)}(1+\varepsilon)^{-\alpha}$. For $A_{i,j}^{(2)}$ write

$$
\begin{aligned}
A_{i,j}^{(2)}(t) &= P\big(Q_{1,i} > (1+\varepsilon)t, Q_{1,i} + M_{1,i}Y_j \le t; Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \\
&\quad + P\big(Q_{1,i} > (1+\varepsilon)t, Q_{1,i} + M_{1,i}Y_j \le t; Y_j \ge -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \\
&\le P\big(M_{1,i}Y_j \le -\varepsilon t, Y_j \ge -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) + P\big(Q_{1,i} > (1+\varepsilon)t, Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \\
&\le P\big(M_{1,i} \ge t^{\frac{\alpha+\beta}{2\beta}}\big) + P\big(Q_{1,i} > (1+\varepsilon)t\big)P\big(Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big).
\end{aligned}
$$

To obtain a bound on $\frac{A_{i,j}^{(2)}(t)}{t^{-\alpha}L(t)}$ tending to zero uniformly on $i, j \in \mathcal{D}$ as $t \to \infty$, we use (A2), Chebyshev's inequality $P\big(M_{1,i} \ge t^{\frac{\alpha+\beta}{2\beta}}\big) \le E\big(M_{1,i}^\beta\big) \cdot t^{-\frac{\alpha+\beta}{2}}$, and the inequality $P\big(Y_j < -\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big) \le C\big(\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big)^{-\alpha}L\big(\varepsilon t^{\frac{\beta-\alpha}{2\beta}}\big)$, which is true for some $C > 0$ in virtue of condition (ii) of the proposition. A uniform bound on $\frac{A_{i,j}^{(3)}(t)}{t^{-\alpha}L(t)}$ which tends to 0 as $\varepsilon \to 0$ follows directly from (A1). Finally, denote $g_{j,t}(a, b) = P\big(Y_j > a^{-1}(t - b)\big)$, fix constants $m \ge 1$ and $n \ge 1$, and let $A_{i,j}^{(4)}(t) = A_{i,j}^{(4,1)}(t) + A_{i,j}^{(4,2)}(t) + A_{i,j}^{(4,3)}(t)$, where

$$
\begin{aligned}
A_{i,j}^{(4,1)}(t) &= E\big(g_{j,t}(M_{1,i}, Q_{1,i})\mathbf{I}_{\{Q_{1,i} \le (1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i} > m\}}\big) \\
A_{i,j}^{(4,2)}(t) &= E\big(g_{j,t}(M_{1,i}, Q_{1,i})\mathbf{I}_{\{Q_{1,i} \le (1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i} \le m\}}\mathbf{I}_{\{|Q_{1,i}| > n\}}\big) \\
A_{i,j}^{(4,3)}(t) &= E\big(g_{j,t}(M_{1,i}, Q_{1,i})\mathbf{I}_{\{Q_{1,i} \le (1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i} \le m\}}\mathbf{I}_{\{|Q_{1,i}| \le n\}}\big).
\end{aligned}
$$

Note that $\frac{A_{i,j}^{(4,1)}(t)}{t^{-\alpha}L(t)} \le E\Big(\frac{g_{j,\varepsilon t}(M_{1,i}, 0)}{(M_{1,i}^{-1}\varepsilon t)^{-\alpha}L(M_{1,i}^{-1}\varepsilon t)} \frac{(M_{1,i}^{-1}\varepsilon t)^{-\alpha}L(M_{1,i}^{-1}\varepsilon t)}{t^{-\alpha}L(t)}\mathbf{I}_{\{M_{1,i} > m\}}\Big)$. Hence, by condition (ii)-(b) of the proposition, $\frac{A_{i,j}^{(4,1)}(t)}{t^{-\alpha}L(t)} \le E\big(\frac{CM_{1,i}^\alpha}{\varepsilon^\alpha} \cdot \frac{L(M_{1,i}^{-1}\varepsilon t)}{L(t)}\mathbf{I}_{\{M_i > m\}}\big)$ for some constant $C > 0$. By (Grey, 1994, Lemma 1), for any $\delta > 0$ there is $K = K(\delta) > 0$ such that $\sup_{t>0} \frac{L(\lambda t)}{L(t)} \le \max\{\lambda^\alpha, K\lambda^{-\delta}\}$ for all $\lambda > 0$. Using $\lambda = M_{1,i}^{-1}\varepsilon$ and $\delta = \frac{\beta-\alpha}{2}$, we obtain

$$
\begin{aligned}
\frac{A_{i,j}^{(4,1)}(t)}{t^{-\alpha}L(t)} &\le E\big(CM_{1,i}^\alpha \varepsilon^{-\alpha} \cdot \max\{(M_{1,i}^{-1}\varepsilon)^\alpha, K(M_{1,i}^{-1}\varepsilon)^{-\frac{\beta-\alpha}{2}}\}\mathbf{I}_{\{M_{1,i} > m\}}\big) \\
&\le E\big(CM_{1,i}^\alpha \varepsilon^{-\alpha}(M_{1,i}^{-\alpha}\varepsilon^\alpha + KM_{1,i}^{\frac{\beta-\alpha}{2}}\varepsilon^{-\frac{\beta-\alpha}{2}})\mathbf{I}_{\{M_{1,i} > m\}}\big).
\end{aligned}
$$

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Hölder's inequality with $p = \frac{2\beta}{\alpha+\beta}$, $q = \frac{2\beta}{\beta-\alpha}$ yields

$$
\begin{aligned}
\frac{A_{i,j}^{(4,1)}(t)}{t^{-\alpha}L(t)} &\leq C\varepsilon^{-\frac{\alpha+\beta}{2}} E\Big[\big(1 + KM_{1,i}^{\frac{\beta+\alpha}{2}}\big)\mathbf{I}_{\{M_{1,i}>m\}}\Big] \\
&\leq C\varepsilon^{-\frac{\alpha+\beta}{2}} E\Big[\big(1 + KM_{1,i}^{\frac{\beta+\alpha}{2}}\big)^{\frac{2\beta}{\alpha+\beta}}\Big]^{\frac{\alpha+\beta}{2\beta}} P(M_{1,i}>m)^{\frac{\beta-\alpha}{2\beta}} \\
&\leq C\varepsilon^{-\frac{\alpha+\beta}{2}} E\Big[\big(1 + K^{\frac{2\beta}{\beta+\alpha}}M_{1,i}^{\beta}\big)\cdot 2^{\frac{\beta-\alpha}{\alpha+\beta}}\Big]^{\frac{\alpha+\beta}{2\beta}} P(M_{1,i}>m)^{\frac{\beta-\alpha}{2\beta}},
\end{aligned}
$$

where we used the inequality $(x+y)^p \leq 2^{p-1}(x^p + y^p)$ which is valid for $x, y \geq 0$ and $p > 1$. Since $P(M_{1,i} > m) \leq m^{-\beta}E\big(M_{1,i}^{\beta}\big)$, it follows from (A2) that $\frac{A_{i,j}^{(4,1)}(t)}{t^{-\alpha}L(t)}$ is uniformly bounded by a function of m which tends to zero when $m \to \infty$. The term $A_{i,j}^{(4,2)}(t)$ is treated similarly, and we therefore omit the details. We next show the asymptotic of $A_{i,j}^{(4,3)}(t)$. If $Q_{1,i} \leq (1-\varepsilon)t$ and $M_{1,i} \leq m$, then $M_{1,i}^{-1}(t - Q_{1,i}) \geq m^{-1}\varepsilon t$. Hence, in virtue of (A1) and condition (ii) of the proposition, we have $P - $ a.s., uniformly on $i, j \in \mathcal{D}$,

$$
\frac{g_{j,t}(M_{1,i}, Q_{1,i})\mathbf{I}_{\{Q_{1,i}\leq(1-\varepsilon)t\}}\mathbf{I}_{\{M_{1,i}\leq m\}}}{\big(M_{1,i}^{-1}(t-Q_{1,i})\big)^{-\alpha}L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)} \to_{t\to\infty} c_j^{(1)}\mathbf{I}_{\{M_{1,i}\leq m\}}. \tag{7}
$$

Furthermore, with probability one, uniformly on $i \in \mathcal{D}$,

$$
t^{\alpha}\big(M_{1,i}^{-1}(t-Q_{1,i})\big)^{-\alpha}\mathbf{I}_{\{M_{1,i}\leq m, |Q_{1,i}|\leq n\}} \to_{t\to\infty} M_{1,i}^{\alpha}\mathbf{I}_{\{M_{1,i}\leq m, |Q_{1,i}|\leq n\}}. \tag{8}
$$

By (Bingham et al., 1987, Theorem 1.2.1), we have a.s., uniformly on $i \in \mathcal{D}$,

$$
\frac{L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)}{L(t)}\mathbf{I}_{\{m^{-1}<M_{1,i}\leq m, |Q_{1,i}|\leq n\}} \to_{t\to\infty} \mathbf{I}_{\{m^{-1}<M_{1,i}\leq m, |Q_{1,i}|\leq n\}}. \tag{9}
$$

By (Bingham et al., 1987, Theorem 1.5.6), $\forall\, \delta > 0\ \exists\, t_0 = t_0(\delta)$ such that

$$
\frac{L\big(M_{1,i}^{-1}(t-Q_{1,i})\big)}{L(t)}\mathbf{I}_{\{M_{1,i}\leq m^{-1}, |Q_{1,i}|\leq n\}} \leq \frac{mt}{t-n}\cdot\mathbf{I}_{\{M_{1,i}\leq m^{-1}\}}, \quad t > t_0. \tag{10}
$$

Estimates (7)-(10) along with assumption (A4) and the bounded convergence theorem show that $\lim_{t\to\infty}\frac{A_{i,j}^{(4)}(t)}{t^{-\alpha}L(t)} = c_j^{(1)}E\big(M_{1,i}^{\alpha}\big)$ uniformly on i, j. $\qquad\square$

To enable us to use Proposition 2.1 iteratively we need the following:

Lemma 2.2. *Let Y satisfy the conditions of Proposition 2.1, and let $\widetilde{Y} = Q_1 + M_1 Y$. Then $\widetilde{Y}$ satisfies condition (ii) of the proposition.*

Proof. Apply Proposition 2.1 to (Y, Q_1, M_1) and $(-Y, -Q_1, M_1)$. $\qquad\square$

The next technical lemma is immediate from Proposition 2.1 and (A3).

Lemma 2.3. *$\exists$ a random variable $Z \geq 0$ satisfying the conditions of Proposition 2.1, s.t. $P(Q_1 + M_1 Z > t|X_1 = i) \leq P(Z > t|X_0 = i)$, $t > 0$, $i \in \mathcal{D}$.*

We are now in position to complete the proof of Theorem 1.3.

Lemma 2.4. *For all $i \in \mathcal{D}$, $\limsup_{t \to \infty} \dfrac{P(R > t | X_0 = i)}{t^{-\alpha}L(t)} \leq (I - G)^{-1}\mathbf{q}^{(1)}(i)$.*

Proof. Let $R_0 = Z$, where Z is as in Lemma 2.3. Then, for all $t > 0$ and $i \in \mathcal{D}$, we have $P(R_1 > t | X_1 = i) \leq P(R_0 > t | X_0 = i)$. This yields:

$$P(R_2 > t | X_2 = i) = \sum_{j \in \mathcal{D}} P(Q_2 + M_2 R_1 > t | X_2 = i, X_1 = j) H(i,j)$$

$$= \sum_{j \in \mathcal{D}} P(Q_{2,i} + M_{2,i} R_1 > t | X_1 = j) H(i,j)$$

$$\leq \sum_{j \in \mathcal{D}} P(Q_{1,i} + M_{1,i} R_0 | X_0 = j) H(i,j) = P(R_1 > t | X_1 = i).$$

Therefore $P(R_2 > t | X_2 = i) \leq P(R_1 > t | X_1 = i)$. Iterating, we obtain

$$P(R_n > t | X_n = i) \leq P(R_{n-1} > t | X_{n-1} = i), \quad \forall\, n \in \mathbb{N}, i \in \mathcal{D}, t > 0. \tag{11}$$

By Proposition 2.1, $\frac{P(R_n > t | X_n = i)}{t^{-\alpha}L(t)} \sim \left[\mathbf{q}^{(1)} + \cdots + G^{n-1}\mathbf{q}^{(1)} + c^* G^n \mathbf{1}\right](i)$, where $c^* > 0$ is a constant such that $P(Z > t) \sim c^* L(t) t^{-\alpha}$ and $\mathbf{1} \in \mathbb{R}^{\mathcal{D}}$ has all components equal to 1. Since $P(Z > 0) = 1$, it follows from (11) that $P(R_n > t | X_n = i) \geq P(R > t | X_0 = i)$ for $n \geq 0$. Hence, $\limsup_{t \to \infty} \dfrac{P(R > t | X_0 = i)}{t^{-\alpha}L(t)} \leq (I - G)^{-1}\mathbf{q}^{(1)}(i)$ for all $i \in \mathcal{D}$. $\qquad\square$

Lemma 2.5. *For all $i \in \mathcal{D}$, $\liminf_{t \to \infty} \dfrac{P(R > t | X_0 = i)}{t^{-\alpha}L(t)} \geq (I - G)^{-1}\mathbf{q}^{(1)}(i)$.*

Proof. Let $\mathcal{R} = Q_{-1} + M_{-1}Q_{-2} + M_{-1}M_{-2}Q_{-3} + \ldots$. Let $R_0 \geq 0$ be independent of $(X_n, Q_{i,n}, M_{i,n})_{n \geq 1, i \in \mathcal{D}}$, s. t. $P(R_0 > t) = P(\mathcal{R} > 0, Q_0 > t)$ for $t > 0$. Then $P(R_0 > t | X_0 = i) \leq P(Q_0 + M_0 \mathcal{R} > t | X_0 = i) = P(R > t | X_0 = i)$. We will now use induction to show that for $n \geq 0$,

$$P(R_n > t | X_n = i) \leq P(R > t | X_0 = i) \qquad \forall\, t > 0, i \in \mathcal{D}. \tag{12}$$

Specifically, assuming (12) for some $n \geq 0$ we obtain

$$P(R_{n+1} > t | X_{n+1} = i) = \sum_{j \in \mathcal{D}} P(Q_{n+1,i} + M_{n=1,i} R_n > t | X_n = j) H(i,j)$$

$$\leq \sum_{j \in \mathcal{D}} P(Q_{1,i} + M_{1,i} R > t | X_0 = j) H(i,j)$$

$$= P(Q_1 + M_1 R > t | X_1 = i) = P(R > t | X_0 = i),$$

Moroever, uniformly on $i \in \mathcal{D}$, $\frac{P(R_0 > t | X_0 = i)}{t^{-\alpha}L(t)} = \frac{P(Q_{0,i} > t) P(\mathcal{R} > 0 | X_0 = i)}{t^{-\alpha}L(t)} \sim e_i$, where $e_i = q_i^{(1)} P(\mathcal{R} > 0 | X_0 = i)$. Let $\mathbf{e} = (e_i)_{i \in \mathcal{D}}$. Then, by Proposition 2.1, $P(R_n > t | X_n = i) \sim t^{-\alpha}L(t) \cdot \left[\mathbf{q}^{(1)} + G\mathbf{q}^{(1)} + \ldots + G^{n-1}\mathbf{q}^{(1)} + G^n \mathbf{e}\right](i)$. This completes the proof of Lemma 2.5 in view of (12). $\qquad\square$

3. Proof of Theorem 1.4

The following result extends Lemma 4 of (Grey, 1994).

Lemma 3.1. *Let* $Y \in \sigma\big(X_n, Q_{n,i}, M_{n,i} : n \leq 0, i \in \mathcal{D}\big)$ *be a random variable s. t.*
$c_i^{(\eta)} := \limsup\limits_{t \to \infty} \frac{P(Y \cdot \eta > t | X_0 = i)}{t^{-\alpha} L(t)}$ *and* $d_i^{(\eta)} := \liminf\limits_{t \to \infty} \frac{P(Y \cdot \eta > t | X_0 = i)}{t^{-\alpha} L(t)}$ *are finite for all* $i \in \mathcal{D}$ *and*
$\eta \in \{-1, 1\}$. *Then for all* $i \in \mathcal{D}, \eta \in \{-1, 1\}$,

$$\limsup_{t \to \infty} \frac{P\big((Q_1 + M_1 Y) \cdot \eta > t \big| X_1 = i\big)}{t^{-\alpha} L(t)} \leq q_i^{(\eta)} + \sum_{j \in \mathcal{D}} \sum_{\gamma \in \{-1, 1\}} G_\gamma(i, j) c_j^{(\gamma)} \quad and$$

$$\liminf_{t \to \infty} \frac{P\big((Q_1 + M_1 Y) \cdot \eta > t \big| X_1 = i\big)}{t^{-\alpha} L(t)} \geq q_i^{(\eta)} + \sum_{j \in \mathcal{D}} \sum_{\gamma \in \{-1, 1\}} G_\gamma(i, j) d_j^{(\gamma)}.$$

Proof. Let $(Y_j)_{j \in \mathcal{D}}$ be random variables independent of both $(X_n)_{n \in \mathbb{Z}}$ and $(Q_{1,i}, M_{1,i})_{i \in \mathcal{D}}$, such that $P(Y_j \cdot \eta > t) = P(Y \cdot \eta > t | X_0 = j)$ for $\eta \in \{-1, 1\}$. Then we have $P\big((Q_1 + M_1 Y) \cdot \eta > t | X_1 = i\big) = \sum_{j \in \mathcal{D}} P\big((Q_{1,i} + M_{1,i} Y_j) \cdot \eta > t\big) H(i, j)$ according ro (5). To complete the proof, apply (Grey, 1994, Lemma 4) separately to each term $P\big((Q_{1,i} + M_{1,i} Y_j) \cdot \eta > t\big)$. $\square$

Let $R^* = |Q_0| + \sum_{n=1}^{\infty} |Q_{-n}| \prod_{i=0}^{n-1} |M_{-i}|$ be a stationary solution of the equation $R_{n+1} = |Q_n| + |M_n| R_n$. Notice that $-R^*$ is a stationary solution of the equation $R_{n+1} = -|Q_n| + |M_n| R_n$. Since $P(-R^* \leq R \leq R^*) = 1$, Theorem 1.3 ensures that Lemma 3.1 can be applied with $Y = R$. Let $a_i^{(\eta)} = \limsup_{t \to \infty} \frac{P(R \cdot \eta > t | X_0 = i)}{t^{-\alpha} L(t)}$ and $b_i^{(\eta)} = \liminf_{t \to \infty} \frac{P(R \cdot \eta > t | X_0 = i)}{t^{-\alpha} L(t)}$. Denote $\mathbf{a}^{(\eta)} = \big(a_i^{(\eta)}\big)_{i \in \mathcal{D}}$, $\mathbf{b}^{(\eta)} = \big(b_i^{(\eta)}\big)_{i \in \mathcal{D}}$ and $\mathbf{a} = \mathbf{a}^{(-1)} + \mathbf{a}^{(1)}$, $\mathbf{b} = \mathbf{b}^{(-1)} + \mathbf{b}^{(1)}$. The application of Lemma 3.1 yields $\mathbf{a} \leq \mathbf{q} + G\mathbf{a}$ and $\mathbf{b} \geq \mathbf{q} + G\mathbf{b}$, which implies $(I - G)^{-1}\mathbf{q} \leq \mathbf{b} \leq \mathbf{a} \leq (I - G)^{-1}\mathbf{q}$. This is only possible if $\mathbf{a}^{(\eta)} = \mathbf{b}^{(\eta)}$, the inequalities in the conclusions of Lemma 3.1 are actually equalities, and thus $\mathbf{a}^{(\eta)} = \mathbf{q}^{(\eta)} + G_1 \mathbf{a}^{(\eta)} + G_{-1} \mathbf{a}^{(-\eta)}$, which implies the result of Theorem 1.4.

4. Proof of Theorem 1.6

Consider the **backward** chain $Z_n = Y_{-n}$, $n \in \mathbb{Z}$, with transition matrix $H(x, y) = P(Y_n = y | Y_{n+1} = x) = \frac{P(Y_0 = y)}{P(Y_0 = x)} \cdot P(Y_{n+1} = y | Y_n = x)$. Fix $y^* \in \mathcal{S}$ and a constant $r \in (0, 1)$. Let $(\eta_n)_{n \in \mathbb{Z}}$ be a sequence of i.i.d. "coins" independent of both $(Z_n)_{n \in \mathbb{Z}}$ and $\big(Q_{n,i}, M_{n,i}\big)_{n \in \mathbb{Z}, i \in \mathcal{D}}$, such that $P(\eta_0 = 1) = r$, $P(\eta_0 = 0) = 1 - r$. Let $N_0 = 0$ and $N_i = \inf\{n > N_{i-1} : Z_n = y^*, \eta_n = 1\}$, $i \in \mathbb{N}$. The blocks $(Z_{N_i}, \ldots, Z_{N_{i+1}-1})$ are independent for $i \geq 0$ and identically distributed for $i \geq 1$. Between two successive regeneration times N_i the chain $(Z_n)_{n \geq 0}$ evolves according to a sub-Markov kernel Θ given by $H(x, y) = \Theta(x, y) + r\mathbf{I}_{\{y = y^*\}} H(x, y)$, i.e.

$$\Theta(x, y) = P(Z_1 = y, N_1 > 1 | Z_0 = x). \tag{13}$$

Theorem A implies that $E\big(e^{\beta N_1} \big| Z_0\big)$ is uniformly bounded for some $\beta > 0$ (see the paragraph following Theorem 1 in (Lalley, 1986)). For $i \geq 0$, let

$$
\begin{aligned}
A_i &= Q_{N_i} + Q_{N_i + 1} M_{N_i} + \ldots + Q_{N_{i+1}-1} M_{N_i} M_{N_i+1} \ldots M_{N_{i+1}-2} \\
B_i &= M_{N_i} M_{N_i+1} \ldots M_{N_{i+1}-1}.
\end{aligned}
$$

The pairs (A_i, B_i) are independent for $i \geq 0$, identically distributed for $i \geq 1$ and $R = A_0 + \sum_{n=1}^{\infty} A_n \prod_{i=0}^{n-1} B_i$. To prove Theorem 1.6 we will verify the conditions of the following theorem for $(A_i, B_i)_{i \geq 1}$.

Theorem B. *(Kesten, 1973; Goldie, 1991) Let $(A_i, B_i)_{i \geq 1}$ be i.i.d. and*

(i) For some $\alpha > 0$, $E\left(|A_1|^{\alpha}\right) = 1$ and $E\left(|B_1|^{\alpha} \log^+ |B_1|\right) < \infty$.

(ii) $P(\log|B_1| = \delta \cdot k$ for some $k \in \mathbb{Z}|B_1 \neq 0) < 1$ for all $\delta > 0$.
    Let $\widetilde{R} = A_1 + \sum_{n=2}^{\infty} A_n \prod_{i=1}^{n-1} B_i$. Then

(a) $\lim_{t \to \infty} t^{\alpha} P\left(\widetilde{R} > t\right) = K_+$, $\lim_{t \to \infty} t^{\alpha} P\left(\widetilde{R} < -t\right) = K_-$ for some $K_+, K_- \geq 0$.

(b) If $P(B_1 < 0) > 0$, then $K_+ = K_-$.

(c) $K_+ + K_- > 0$ if and only if $P\left(A_1 = (1 - B_1)c\right) < 1$ for all $c \in \mathbb{R}$.

For $\beta \geq 0$ define matrices H_{β} and Θ_{β} by setting $H_{\beta}(x, y) = H(x, y)E(|M_{0,\zeta(y)}|^{\beta})$ and $\Theta_{\beta}(x, y) = \Theta(x, y)E(|M_{0,\zeta(y)}|^{\beta})$. Since we have $E\left(\prod_{i=0}^{n} |M_{-i}|^{\beta} \mathbf{I}_{\{Z_n=y\}} \middle| Z_0 = x\right) = E\left(|M_0|^{\beta} \middle| Z_0 = x\right) H_{\beta}^n(x, y)$ and, for $y \neq y^*$,

$$E\left(\mathbf{I}_{\{n < N_1, Z_n = y\}} \prod_{i=0}^{n} |M_{-i}|^{\beta} \middle| Z_0 = x\right) = E\left(|M_0|^{\beta} \middle| Z_0 = x\right) \Theta_{\beta}^n(x, y),$$

(Roitershtein, 2007, Lemma 2.3 and Proposition 2.4) show that (4) holds and that the spectral radius of H_{α} is 1, while the spectral radius of Θ_{α} is less than 1. Hence, see (2.43) in (Mayer-Wolf et al., 2004), we have:

Lemma 4.1. $E(|B_1|^{\alpha}) = 1$.

We next show that $\log|B_0|$ is a non-lattice random variable if $y^* \in \mathcal{D}$.

Lemma 4.2. *Assume $y^* \in \mathcal{D}^1$ and let $V = \sum_{n=0}^{N_1 - 1} \log|M_{-n}| = \log|B_0|$. Then for any $\delta > 0$ we have $P\left(V \in \delta \cdot \mathbb{Z} \middle| Z_0 = y^*\right) < 1$, where $\delta \cdot \mathbb{Z} := \{k \cdot \delta : k \in \mathbb{Z}\}$.*

Proof. For $y \in \mathcal{D}$ and $k \in \mathbb{N}$, let $y_{(k)}$ denote $(y_1, \dots, y_k) \in \mathcal{D}^k$ such that $y_2 = \dots = y_k = y$ and $y_1 = y^*$. By Lalley's Theorem A, for any $y \in \mathcal{D}$,

$$P(Z_0 = y_{(1)}, Z_1 = y_{(m)}, \dots, Z_{m-1} = Z_{N_1 - 1} = y_{(2)}) > 0 \tag{14}$$

for some $m > 1$. Therefore, $P\left(V \in \delta \cdot \mathbb{Z} | Z_0 = (y^*)\right) = 1$ along with (14) imply: (i) using $y = y^*$, $P\left(m \cdot \log|M_{0,y^*}| \in \delta \cdot \mathbb{Z} | Z_0 = (y^*)\right) = 1$; (ii) using general $y \in \mathcal{D}$, $P\left((m-1) \cdot \log|M_{0,y}| + \log|M_{0,y^*}| \in \delta \cdot \mathbb{Z} | Z_0 = (y^*)\right) = 1$. Therefore, we would have $P\left(m(m-1) \cdot \log|M_{0,y}| \in \delta \cdot \mathbb{Z} | Z_0 = (y^*)\right) = 1$ for any $y \in \mathcal{D}$, contradicting condition (iii) of Theorem 1.6. $\square$

The proof of the next lemma is verbatim the proof of (2.44) in (Mayer-Wolf et al., 2004), and therefore is omitted.

Lemma 4.3. $\exists \beta > \alpha$ *s. t.* $E\left[\left(\sum_{n=0}^{N_1 - 1} \prod_{i=0}^{n-1} |M_{-i}|\right)^{\beta} \middle| Z_0\right]$ *is bounded a.s.*

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Completion of the proof of Theorem 1.6.

(a)–(d) It follows from Lemmas 4.1–4.3 that the conclusions of Theorem B can be applied to (A_1, B_1). Claims (a) through (d) of Theorem 1.6 follow then from the identity $R = A_0 + B_0\widetilde{R}$ and the independence of (A_0, B_0) of $\widetilde{R}$ under the conditional measures $P(\cdot|Z_0 = x)$. In particular, for $\eta \in \{-1, 1\}$,

$$\lim_{t\to\infty} P\big(R \cdot \eta > t|Z_0\big) = E\big(|B_0|^\alpha(\mathbf{I}_{\{B_0\cdot\eta>0\}}K_+ + \mathbf{I}_{\{B_0\cdot\eta<0\}}K_-)\big|Z_0\big) \tag{15}$$

Note that Theorem A-(iv) implies $P(N_1 \text{ is odd}) > 0$ and $P(N_1 \text{ is even}) > 0$. Therefore, $P(M_0 < 0) > 0$ yields $P(B_0 < 0|Z_0 = x) > 0$ for all $x \in \mathcal{S}$.

(e) First, (3) implies $P\big(R_n = \Gamma(X_n)\big) = 1$ provided that $R_0 = \Gamma(X_0)$. Hence if (3) is true and $R_0 = \Gamma(X_0)$, then R_n can take only a finite number of values from $\{\Gamma(i) : i \in \mathcal{D}\}$ and cannot converge in distribution to a power-tailed random variable. Therefore, (3) implies $P\big(K_1(\mathcal{X}) + K_{-1}(\mathcal{X}) = 0\big) = 1$.

Assume now $P\big(K_1(\mathcal{X}) + K_{-1}(\mathcal{X}) = 0\big) = 1$. Then (15) and Theorem 1.6-(b) imply $K_+ = K_- = 0$. Then, by Theorem B, $P\big(A_1 + c(y^*)B_1 = c(y^*)\big) = 1$ for a constant $c(y^*)$. Hence $P\big(Q_0 + M_0\widehat{R} = c(y^*)\big|Z_0 = y^*\big) = 1$, with $\widehat{R} = \frac{R-Q_0}{M_0}$. Since $\widehat{R}$ is conditionally independent of (Q_0, M_0) given Z_0,

$$P\big(\widehat{R} = c_1(y^*)\big|Z_0 = y^*\big) = P\big(R = c_1(y^*)\big|Z_{-1} = y^*\big) = 1, \tag{16}$$

$$P\big(Q_{0,\zeta(y^*)} + M_{0,\zeta(y^*)}c_1(y^*) = c(y^*)\big) = 1, \tag{17}$$

for some constant $c_1(y^*)$. It follows from (16) and the Markov property that $P\big(R = c_1(y^*)\big|Z_0 = y\big) = 1 \ \forall \ y \in \mathcal{S}$ s. t. $P(Z_0 = y|Z_{-1} = y^*) > 0$. Then

$$P\big(R = c(y^*)\big|Z_{-1} = x\big) = 1 \ \forall \ x \in \mathcal{S} \text{ s. t. } P(Z_0 = y^*|Z_{-1} = x) > 0. \tag{18}$$

Thus $P\big(R = c_1(x) = c(y^*)\big|Z_{-1} = x\big) = 1$ whence $P(Z_0 = y^*|Z_{-1} = x) > 0$. Since $y^* \in \mathcal{S}$ is arbitrary, $P\big(Q_{0,\zeta(Z_0)} + M_{0,\zeta(Z_0)}c_1(Z_0) = c_1(Z_{-1})\big) = 1$ due to (17). This is exactly (3) once one sets $\Gamma(Y_n) = c_1(Z_{-n})$.

Suppose now in addition that $(Q_{0,i}, M_{0,i})$, $i \in \mathcal{D}$, are non-degenerate. Since the system $q + mc_1 = c$, $q + m\tilde{c}_1 = \tilde{c}$ for unknown variables q, m has a unique solution unless $c_1 = \tilde{c}_1$, it follows from (17) that c_1 depends on $\zeta(y^*)$ only. Let $\Gamma(i) = c_1(y^*)$ for $i = \zeta(y^*)$. Then, (17) and (18) imply $P\big(Q_0 + M_0\Gamma(X_0) = \Gamma(X_1)\big) = 1$. Hence, in virtue of (i) of Definition 1.5, $P(\Gamma(X_n) = c) = P\big(Q_0 + M_0c = c\big) = 1$ for some constant $c \in \mathbb{R}$.

References

Berbee, H., 1987. Chains with infinite connections: Uniqueness and Markov representation. Probab. Theory Rel. Fields 76, 243–253.

Bingham, N.H., Goldie, C.M., Teugels, J.L., 1987. Regular Variation. Encyclopedia of Mathematics and its Applications, Cambridge University Press.

Bowen, R., 1975. Equillibrium States and the Ergodic Anosov Diffeomorthisms. *Lecture Notes in Mathematics 470.* Springer-Verlag.

Brandt, A., 1986. The stochastic equation $Y_{n+1} = A_nY_n + B_n$ with stationary coefficients. Adv. in Appl. Probab. 18, 211–220.

Collamore, J.F., 2009. Random recurrence equations and ruin in a Markov-dependent stochastic economic environment. Ann. Appl. Probab. 19, 1404–1458.

Davis, R.A., Hsing, T., 1995. Point process and partial sum convergence for weakly dependent random variables with infinite variance. Ann. Probab 23, 879–917.

Embrechts, P., Goldie, C.M., 1994. Perpetuities and random equations, in: Mandl, P., Hušková, M. (Eds.), Asymptotic Statistics, 5th. Symp. (Prague, 1993), Contrib. Statist. Physica, Heidelberg, pp. 75–86.

Ephraim, Y., Merhav, N., 2002. Hidden Markov processes. IEEE Trans. Inform. Theory 48, 1518–1569.

Fernández, R., Maillard, G., 2004. Chains with complete connections and one-dimensional Gibbs measures. Electron. Journ Probab. 9, 145–176.

Goldie, C.M., 1991. Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab. 1, 126–166.

Grey, D.R., 1994. Regular variation in the tail of solutions of random difference equations. Ann. Appl. Probab. 4, 169–183.

Grincevičius, A.K., 1975. One limit distribution for a random walk on the line. Lithuanian Math. J. 15, 580–589.

de Haan, L., Resnick, S.I., Rootzén, H., de Vries, C.G., 1989. Extremal behavior of solutions to a stochastic difference equation with applications to ARCH processes. Stochastic Process. Appl. 32, 213–224.

Iosifesku, M., Grigoresku, S., 1990. Dependence with Complete Connections and its Applications. *Cambridge Tracts in Mathematics 96*. Cambridge.

Kaijser, T., 1981. On a new contraction condition for random systems with complete connections. Rev. Roumaine Math. Pures Appl. 26, 1075–1117.

Keane, M., 1972. Strongly mixing *g*-measures. Invent. Math. 16, 309–324.

Kesten, H., 1973. Random difference equations and renewal theory for products of random matrices. Acta. Math. 131, 208–248.

Konstantinides, D., Mikosch, T., 2005. Large deviations for solutions to stochastic recurrence equations with heavy-tailed innovations. Ann. Prob. 33, 1992–2035.

Lalley, S., 1986. Regenerative representation for one-dimensional Gibbs states. Ann. Probab. 14, 1262–1271.

Mayer-Wolf, E., Roitershtein, A., Zeitouni, O., 2004. Limit theorems for one-dimensional transient random walks in Markov environments. Ann. Inst. H. Poincaré-Probab. Statist. 40, 635–659.

Perrakis, S., Henin, C., 1974. The evaluation of risky investments with random timing of cash returns. Management Sci. 21, 79–86.

Rachev, S., Samorodnitsky, G., 1995. Limit laws for a stochastic process and random recursion arising in probabilistic modeling. Adv. Appl. Prob. 27, 185–202.

Rastegar, R., Roytershteyn, V., Roitershtein, A., Suh, J., 2010. Discrete-time Langevin motion of a particle in a Gibbsian random potential. Preprint.

Resnick, S.I., Subramanian, A., 1998. Heavy tailed hidden semi-Markov models. Stoch. Models 14, 319–334.

Roitershtein, A., 2007. One-dimensional linear recursions with Markov-dependent coefficients. Ann. Appl. Probab. 17, 572–608.

de Saporta, B., 2005. Tails of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ with Markovian coefficients. Stoch. Proc. Appl. 115, 1954–1978.

Vervaat, W., 1979. On a stochastic difference equations and a representation of non-negative infinitely divisible random variables. Adv. Appl. Prob. 11, 750–783.