Relative growth of the partial sums of certain random Fibonacci-like sequences

Alexander Roitershtein* Zirou Zhou †

January 19, 2017; Revised August 8, 2017

Abstract

We consider certain Fibonacci-like sequences $(X_n)_{n\geq 0}$ perturbed with a random noise. Our main result is that $\frac{1}{X_n}\sum_{k=0}^{n-1} X_k$ converges in distribution, as n goes to infinity, to a random variable W with Pareto-like distribution tails. We show that $s = \lim_{x\to\infty} \frac{-\log P(W>x)}{\log x}$ is a monotonically decreasing characteristic of the input noise, and hence can serve as a measure of its strength in the model. Heuristically, the heavy-tailed limiting distribution, versus a light-tailed one with $s = +\infty$, can be interpreted as an evidence supporting the idea that the noise is "singular" in the sense that it is "big" even in a "slightly" perturbed sequence.

MSC2010: Primary 60H25, 60J10, secondary 60K20.
Keywords: random linear recursions, tail asymptotic, Lyapunov constant, Markov chains, regeneration structure.

1 Introduction and statement of the main result

Let $(\eta_n)_{n\geq 0}$ be a sequence of independent Bernoulli random variables with $P(\eta_n = 1) = 1-\varepsilon$, $P(\eta_n = 0) = \varepsilon$ for some $\varepsilon \in (0,1)$. We consider a sequence $(X_n)_{n\geq 0}$ of real-valued random variables generated by the recursion

$$X_{n+1} = aX_n + b\eta_{n-1}X_{n-1}, \qquad n \in \mathbb{N}, \tag{1}$$

with the initial conditions $X_0 = 1$, $X_1 = a$, where

$$a \in (0,1) \qquad \text{and} \qquad b > 1-a \tag{2}$$

are given deterministic constants.

The above construction is inspired by the models considered in [1]. The sequence X_n can be thought as a perturbation with noise of its deterministic counterpart, which is defined through the recursion equation

$$Z_{n+1} = aZ_n + bZ_{n-1}$$

*Dept. of Mathematics, Iowa State University, Ames, IA 50011, USA; e-mail: roiterst@iastate.edu
†Dept. of Mathematics, Iowa State University, Ames, IA 50011, USA; e-mail: zzhou@iastate.edu

and the initial conditions $Z_0 = 1$, $Z_1 = a$. Throughout the paper we are interested in the dependence of model's characteristics on the parameter ε that varies while the recursion coefficients a, b are maintained fixed.

It is not hard to check that $\lim_{n\to\infty} \frac{1}{Z_n}\sum_{k=0}^{n-1} Z_k = (\lambda_1 - 1)^{-1}$, where λ_1 is a constant defined below in (8). In this paper we are primarily concerned with the asymptotic behavior of the following sequence

$$W_n := \frac{1}{X_n}\sum_{k=0}^{n-1} X_k, \qquad n \in \mathbb{N}, \tag{3}$$

which describes the rate of growth of the partial sums relatively to the original sequence X_n. Our main result is stated in the following theorem. Intuitively, it can be interpreted as a saying that while adding more noise to the input by increasing the value of ε yields more noise in the output sequence W_n, the noise remains large for all, even arbitrarily small, values of the parameter $\varepsilon > 0$ in some rigorous sense.

Theorem 1. *Let W_n be defined in (3). Then the following holds true:*

(a) There exists $\varepsilon^* \in (0, 1)$ such that

(i) If $\varepsilon \in (0, \varepsilon^*)$, then W_n converges in distribution, as n goes to infinity, to a non-degenerate random variable $W^{(\varepsilon)}$.

(ii) If $\varepsilon \in [\varepsilon^*, 1)$, then $\lim_{n\to\infty} P(W_n > x) = 1$ for any $x > 0$, that is $W^{(\varepsilon)} = +\infty$.

(iii) For any $\varepsilon \in (0, \varepsilon^*)$, there exist reals $s_\varepsilon \in (0,\infty)$ and $K_\varepsilon \in (0,\infty)$ such that

$$\lim_{x\to\infty} P(W^{(\varepsilon)} > x)x^{s_\varepsilon} = K_\varepsilon.$$

(b) Furthermore, s_ε is a continuous strictly decreasing function of ε on $(0, \varepsilon^*)$, and

$$\lim_{\varepsilon\downarrow 0} s_\varepsilon = \infty \qquad \text{while} \qquad \lim_{\varepsilon\uparrow\varepsilon^*} s_\varepsilon = 0. \tag{4}$$

The specific choice of the initial values $X_0 = 1$ and $X_1 = a$ is technically convenient, but is not essential. In particular, while asserting it ultimately yields part *(a)* of Lemma 3, changing it wouldn't affect part *(b)* of the lemma. Theorem 1 remains valid for an arbitrary pair (X_0, X_1) of positive numbers. See Remark 9 in Section 3 for details. To extend Theorem 1 to a linear recursion (1) under a more general than (2) assumption $a \neq 0$, $b > 0$, one can consider $\widetilde{X}_n = \theta^{-n}X_n$ with an arbitrary $\theta \in \mathbb{R}$ such that $a\theta > 0$ and $2|a| < 2|\theta| < |a| + \sqrt{a^2 + 4b}$. The new sequence $\widetilde{X}_n$ satisfies the recursion $\widetilde{X}_{n+1} = \widetilde{a}\widetilde{X}_n + \widetilde{b}\eta_{n-1}\widetilde{X}_{n-1}$ with $\widetilde{a} = a/\theta < 1$ and $\widetilde{b} = b/\theta^2 > 1 - \widetilde{a}$. Some other readily available extensions of Theorem 1 are discussed in Section 5 below.

The proof of Theorem 1 is given in Section 3 below. Note that the theorem implies that the limiting distribution $W^{(\varepsilon)}$ has power tails as long as it is finite and non-degenerate. We remark that additional properties of the constants ε^* and s_ε can be inferred from the auxiliary results discusses in Section 3 below. In particular, see Proposition 6 which provides some information on the relation of s_ε to the Lyapunov exponent and the moments of the reciprocal sequence X_n^{-1}.

For an integer $n \geq 0$, let

$$R_n = \frac{X_n}{X_{n+1}}. \tag{5}$$

The sequence R_n forms a Markov chain since (1) is equivalent to $R_n = (a + b\eta_{n-1}R_{n-1})^{-1}$. Notice that, since $X_0 = 1$, for $n \in \mathbb{N}$ we have $X_n^{-1} = \prod_{k=0}^{n-1} R_n$ and

$$W_{n+1} = R_n W_n + R_n \quad \text{or, equivalently,} \quad (W_{n+1} + 1) = R_n(W_n + 1) + 1. \tag{6}$$

The proof of the assertion *(a)-(iii)* of Theorem 1 is carried out by an adaption of the technique used in [11] to obtain an extension of Kesten's theorem [6, 8] for linear recursions with i. i. d. coefficients to a setup with Markov-dependent coefficients. More specifically, to prove that the distribution of $W^{(\varepsilon)}$ is asymptotically power-tailed, we verify in Section 3 that Markov chain R_n satisfies Assumption 1.5 in [11]. This allows us to borrow key auxiliary results from [11, 13] and also use a variation of the underlying regeneration structure argument in [11]. See Lemma 7 in Section 3 below for details.

The proof of Theorem 1 relies in particular on the asymptotic analysis of the negative moments of X_n (more specifically, the function $\Lambda_\varepsilon(t)$ defined below in (18)). First positive integer moments of X_n can be in principle computed explicitly. We conclude this introduction with the statement of a result which is not directly connected to Theorem 1, but might be useful, for instance, for the statistical analysis of the sequence X_n. Here and throughout this paper we use the notation E_P to denote the expectation operator under the probability law P (in order to distinguish it from the expectation E_Q, where Q is introduced in Section 2 below). For $\varepsilon \in [0, 1)$, let

$$\lambda_{\varepsilon,1} = \frac{a + \sqrt{a^2 + 4b(1-\varepsilon)}}{2} > 0 \quad \text{and} \quad \lambda_{\varepsilon,2} = \frac{a - \sqrt{a^2 + 4b(1-\varepsilon)}}{2} < 0 \tag{7}$$

denote the roots of the characteristic equation $\lambda^2 = a\lambda + b(1-\varepsilon)$. We have:

Proposition 2. *For any integer $n \geq 0$,*

(a) We have $E_P(X_n) = \dfrac{\lambda_{\varepsilon,1}^{n+1} - \lambda_{\varepsilon,2}^{n+1}}{\lambda_{\varepsilon,1} - \lambda_{\varepsilon,2}}$. In particular, $\lim_{n\to\infty} \frac{1}{n} \log E_P(X_n) = \lambda_{\varepsilon,1}$.

(b) We have $\lim_{n\to\infty} \frac{1}{n} \log E_P(X_n) = a\lambda_{\varepsilon,1} + b(1-\varepsilon)$. More precisely,

$$E_P(X_n^2) = c_1[a\lambda_{\varepsilon,1} + b(1-\varepsilon)]^n + c_2[a\lambda_{\varepsilon,2} + b(1-\varepsilon)]^n - \frac{2(-b)^{n+1}(1-\varepsilon)^n}{4b(1-\varepsilon) + a^2},$$

where are the constants c_1 and c_2 are chosen in a manner consistent with the initial conditions $X_0 = 1$ and $X_1 = a$.

(c) Letting $U_{n,k} := E_P(X_n X_{n+k})$ for an integer $k \geq 0$,

$$U_{n,k} = d_{n,1}\lambda_{\varepsilon,1}^k + d_{n,2}\lambda_{\varepsilon,2}^k,$$

where are the constants $d_{n,1}$ and $d_{n,2}$ are chosen in a manner consistent with the initial conditions $U_{n,0} = E_P(X_n^2)$ and

$$U_{n,1} = E_P(X_n X_{n+1}) = \frac{1}{a}\big[E_P(X_{n+1}^2) - b(1-\varepsilon)E_P(X_n^2) - (-b)^{n+1}(1-\varepsilon)^n\big].$$

In particular, for any $k \in \mathbb{N}$ we have $\lim_{n\to\infty} \frac{1}{n} \log E_P(X_n X_{n+k}) = a\lambda_{\varepsilon,1} + b(1-\varepsilon)$.

The rest of the paper is organized as follows. Section 2 contains a preliminary discussion and an auxiliary monotonicity result (with respect to the parameter ε) about the Lyapunov constant of the sequence X_n. The proof of Theorem 1 is included in Section 3. The proof of Proposition 2 is deferred to Section 4. Finally, section 5 contains some concluding remarks regarding possible extensions of the results in Theorem 1.

2 Preliminaries: Lyapunov constant of X_n

This section includes a preliminary discussion which is focused on the random variable R_n defined in (5) and the Lyapunov constant $\gamma(\varepsilon)$ introduced below. The main purpose here is to obtain a monotonicity result in Proposition 4. The coupling construction employed to prove Proposition 4 is also used in Section 3, to carry out the proof of Propositions 6 and 11.

Recall $\lambda_{\varepsilon,1}$ and $\lambda_{\varepsilon,2}$ from (7). In order to simplify the notation, denote

$$\lambda_1 := \lambda_{0,1} = \frac{a + \sqrt{a^2+4b}}{2} \qquad \text{and} \qquad \lambda_2 := \lambda_{0,2} = \frac{a - \sqrt{a^2+4b}}{2}. \tag{8}$$

Notice that the condition $a + b > 1$ ensures $\lambda_1 > 1$. Using the initial conditions $Z_0 = 1$ and $Z_1 = a$, one can verify that

$$Z_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \qquad n \geq 0. \tag{9}$$

Using (9) one can obtain a Cassini-type identity $Z_{n-1}Z_{n+1} - Z_n^2 = b^n(-1)^{n+1}$ (see, for instance, Theorem 5.3 in [9] for the original Fibonacci sequence result) and the identity $Z_{n+1} - \lambda_1 Z_n = \lambda_2^{n+1}$. The alternating sign of the right-hand side in these two identities yields for $k \in \mathbb{N}$,

$$a \leq \frac{Z_{2k-1}}{Z_{2k-2}} < \frac{Z_{2k+1}}{Z_{2k}} < \lambda_1 < \frac{Z_{2k+2}}{Z_{2k+1}} < \frac{Z_{2k}}{Z_{2k-1}} \leq \frac{a^2+b}{a}. \tag{10}$$

Recall R_n from (5). The (unique) stationary distribution for the countable, irreducible and aperiodic Markov chain R_n can be obtained as follows. For $n \in \mathbb{N}$, let

$$T_n = \sup\{i \leq n : \eta_i = 0\}. \tag{11}$$

Fix any $k \in \mathbb{N}$. Then for a positive integer $n > k$ we have

$$\begin{aligned} P\big(R_n = Z_{k-1}/Z_k\big) &= P(T_n = n-k-1) \\ &= P\big(\eta_{n-k-1} = 0, \eta_{n-k} = \eta_{n-k+1} = \ldots = \eta_{n-3} = \eta_{n-2} = 1\big) = \varepsilon(1-\varepsilon)^{k-1}. \end{aligned}$$

Thus, $\lim_{n\to\infty} P\big(R_n = Z_{k-1}/Z_k\big) = \varepsilon(1-\varepsilon)^{k-1}$. The stationary sequence $(R_n)_{n\in\mathbb{N}}$ can be extended into a double-infinite stationary sequence $(R_n)_{n\in\mathbb{Z}}$ [4]. Let Q denote the law of the time-reversed stationary Markov chain $(R_{-n})_{n\in\mathbb{Z}}$. We have established the following result:

Lemma 3. *Let $S_k = \frac{Z_{k-1}}{Z_k}$, $k \geq 1$. Then:*

(a) For all $n \in \mathbb{Z}$, we have $P\big(R_n \in \{S_k : k \in \mathbb{N}\}\big) = 1$.

(b) Furthermore, $Q(R_n = S_k) = \varepsilon(1-\varepsilon)^{k-1}$ for any $n \in \mathbb{Z}, k \in \mathbb{N}$.

Let $\gamma = \gamma(\varepsilon)$ denote the Lyapunov exponent of the sequence X_n, that is

$$\gamma := \lim_{n\to\infty} \frac{1}{n}\log X_n = \lim_{n\to\infty}\frac{1}{n}E_P(\log X_n) = E_Q\Big(\log\frac{1}{R_1}\Big), \quad P-\text{a.s. and } Q-\text{a.s.} \tag{12}$$

The existence of the limit along with the identities follow from results in [5]. Taking in account that $Z_0 = 1$, (10) implies that

$$\gamma \;=\; -E_Q(\log R_1) = -\sum_{n=1}^{\infty}\varepsilon(1-\varepsilon)^{n-1}\log S_n = \sum_{n=1}^{\infty}\varepsilon^2(1-\varepsilon)^{n-1}\log Z_n. \tag{13}$$

The last formula can be compactly written as $\gamma = \varepsilon\cdot E_P(\log Z_T)$, where

$$T = 1 + \inf\{k \geq 0 : \eta_k = 0\} = \inf\{j \geq 1 : R_j = 1/a\}. \tag{14}$$

It follows from (13) and the fact that $|\log S_n|$ is a bounded sequence, that $\gamma(\varepsilon)$ is an analytic function of ε on $[0,1]$. In particular,

$$\lim_{\varepsilon\downarrow 0}\gamma(\varepsilon) = \lim_{n\to\infty}\log S_n = \log\lambda_1 \qquad \text{and} \qquad \lim_{\varepsilon\uparrow 1}\gamma(\varepsilon) = \log a. \tag{15}$$

We remark that the analyticity of $\gamma(\varepsilon)$ on $[0,1)$ follows directly from a general result in [12]. For recent advances in numerical study of the Lyapunov exponent for random Fibonacci sequences see [10, 15] and references therein.

We next prove formally the following intuitive result. Together with (15) it implies the existence of $\varepsilon^* \in (0,1)$ such that $\gamma(\varepsilon) > 0$ if and only if $\varepsilon < \varepsilon^*$. Our interest to this phase transition steams from the result in Lemma 5 stated below in Section 3.

Proposition 4. *The function $\gamma(\varepsilon) : [0,1] \to \mathbb{R}$ is strictly decreasing.*

Proof. The proof is by a coupling argument. Fix any $\varepsilon \in [0,1)$ and $\varepsilon_1 \in (\varepsilon, 1]$. Let $(X_n)_{n\geq 0}$ be the sequence introduced in (1), and define $(X_n^{(1)})_{n\geq 0}$ as follows: $X_0^{(1)} = 1$, $X_1^{(1)} = a$, and

$$X_{n+1}^{(1)} = aX_n^{(1)} + b\eta_{n-1}^{(1)}X_{n-1}^{(1)}, \qquad n \in \mathbb{N},$$

where $\eta_n^{(1)} = \min\{\eta_n, \xi_n\}$ and ξ_n are i. i. d. random variables with the distribution

$$\xi_n = \begin{cases} 0 & \text{with probability } \frac{\varepsilon_1-\varepsilon}{1-\varepsilon} \\ 1 & \text{with probability } \frac{1-\varepsilon_1}{1-\varepsilon}, \end{cases}$$

such that ξ_n is independent of the σ-algebra $\sigma(X_0,\eta_0,X_1,\eta_1,\ldots,X_{n-1},\eta_{n-1},X_n,X_{n+1})$ for all $n\geq 0$. Then $P\big(\eta_n^{(1)}=0\big)=\varepsilon_1$, $P\big(\eta_n^{(1)}=1\big)=1-\varepsilon_1$, and hence the sequence $(X_n^{(1)})_{n\geq 0}$ is distributed according to the same law as $(X_n)_{n\geq 0}$ with ε_1 replacing ε in the definition of η_n. To deduce that $\gamma(\varepsilon)$ is a non-increasing function of ε, observe that by the coupling construction, $\eta_n\geq \eta_n^{(1)}$ for all $n\geq 0$, and hence, by induction, $X_n\geq X_n^{(1)}$ for all $n\geq 0$.

To conclude the proof of the proposition it remains to show that $\gamma(\varepsilon)$ is strictly decreasing. Toward this end, first observe that for any integer $n\geq 2$ we have

$$\zeta_n:=\mathbf{1}\big(\eta_n^{(1)}\neq\eta_n\big)=\mathbf{1}\big(\eta_n^{(1)}=0,\eta_n=1\big)=\mathbf{1}\big(\eta_n=1,\xi_n=0\big),$$

where $\mathbf{1}(A)$ denotes the indicator function of the event A and the first equality serves as a definition of ζ_n. Then, the following is an implication of Lemma 3 and (10) along with the fact (which we have established) that $X_n\geq X_n^{(1)}$ for all $n\geq 0$:

$$\begin{aligned}\log X_n-\log X_n^{(1)} &= \log\frac{X_n}{X_n^{(1)}}\geq \log\left[\prod_{k=0}^{n-2}\Big(\frac{aX_{n-1}+bX_{n-2}}{aX_{n-1}^{(1)}}\Big)^{\zeta_k}\right]\\ &\geq \log\left[\prod_{k=0}^{n-2}\Big(\frac{aX_{n-1}+bX_{n-2}}{aX_{n-1}}\Big)^{\zeta_k}\right]\geq \log\left[\prod_{k=0}^{n-2}\Big(1+\frac{b}{a^2+b}\Big)^{\zeta_k}\right]\\ &\geq \sum_{k=0}^{n-2}\zeta_k\cdot\log\Big(1+\frac{b}{a^2+b}\Big). \end{aligned}\tag{16}$$

It follows then from (12) and the law of large numbers that with probability one,

$$\begin{aligned}\gamma(\varepsilon)-\gamma(\varepsilon_1) &= \lim_{n\to\infty}\frac{1}{n}\big(\log X_n-\log X_n^{(1)}\big)\geq \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-2}\zeta_n\cdot\log\Big(1+\frac{b}{a^2+b}\Big)\\ &= E_P(\zeta_n)\cdot\log\Big(1+\frac{b}{a^2+b}\Big)=(\varepsilon_1-\varepsilon)\cdot\log\Big(1+\frac{b}{a^2+b}\Big)>0.\end{aligned}$$

The proof of the proposition is complete. □

3 Proof of the main result

The purpose of this section is to prove Theorem 1. The proof is divided into a sequence of lemmas. The critical exponent ε^* is identified and part *(a)-(i)* and *(a)-(ii)* of the theorem are proved in Proposition 6. The assertion in part *(a)-(iii)* of the theorem is verified in Lemma 7. Finally, the claim in part *(b)* is established in Proposition 11.

Observe that under the stationary law Q the random variable $W_n=\sum_{k=0}^{n-1}\prod_{j=k}^{n-1}R_j$ has the same distribution as $\sum_{k=0}^{n-1}\prod_{j=0}^{k}R_{-j}$. Therefore one can write $W^{(\varepsilon)}=\sum_{k=0}^{\infty}\prod_{j=0}^{k}R_{-j}$. The following lemma is well-known, see for instance Theorem 2.1.2 (especially display (2.1.6)) and the subsequent Remark in [14].

Lemma 5. *For any $\varepsilon\in(0,1)$ we have $P(W^{(\varepsilon)}<\infty)=Q(W^{(\varepsilon)}<\infty)\in\{0,1\}$. Moreover, $P(W^{(\varepsilon)}<\infty)=1$ if and only if $\gamma(\varepsilon)=-E_Q(\log R_1)>0$.*

Let $\varepsilon^* = \sup\{\varepsilon > 0 : \gamma(\varepsilon) > 0\}$. It follows from Proposition 4, the limits in (15), and the continuity of $\gamma(\varepsilon)$ that

$$\varepsilon^* \in (0,1) \qquad \text{and} \qquad \gamma(\varepsilon^*) = 0. \tag{17}$$

By virtue of Lemma 5, ε^* satisfies *(a)-(i)* and *(a)-(ii)* in the statement of Theorem 1. We proceed with the proof that *(a)-(iii)* of the theorem also holds true.

Following [11, 13], we are going to identify the critical exponent s_ε in the statement of Theorem 1 as the unique solution to the equation $\Lambda_\varepsilon(s_\varepsilon) = 0$, where for $t \geq 0$ we define

$$\Lambda_\varepsilon(t) := \lim_{n\to\infty} \frac{1}{n} \log E_Q\big(R_1^t \ldots R_n^t\big).$$

It follows from Lemmas 2.6 and 2.8(a) in [11] (see especially display (2.11) in [11]) applied to the forward Markov chain $(R_n)_{n\geq 0}$ that the above limit exists and in fact is not affected by the initial distribution of the Markov chain. In particular we have:

$$\Lambda_\varepsilon(t) = \lim_{n\to\infty} \frac{1}{n} \log E_P\Big(\frac{X_0^t}{X_{n-1}^t}\Big) = \lim_{n\to\infty} \frac{1}{n} \log E_P\Big(\frac{1}{X_n^t}\Big) \tag{18}$$

The following proposition is a key ingredient in the proof of Theorem 1. Note that for any $\varepsilon \in [0,1]$, $\Lambda_\varepsilon(0) = 0$ and, by virtue of the Cauchy-Schwarz inequality, $\Lambda_\varepsilon(t)$ is convex on $[0,\infty)$. In particular, the one-sided derivative $\Lambda_\varepsilon'(0) := \lim_{t\downarrow 0} \Lambda_\varepsilon(t)/t$ is well-defined.

Proposition 6. *Let $\varepsilon^* \in (0,1)$ be defined in* (17). *Then the following four statements are equivalent for $\varepsilon \in (0,1)$:*

(i) $\gamma(\varepsilon) > 0$, *that is* $\varepsilon \in (0,\varepsilon^*)$.

(ii) $\Lambda_\varepsilon'(0) < 0$.

(iii) There exists a unique $s_\varepsilon > 0$ such that $\Lambda_\varepsilon(s_\varepsilon) = 0$.

(iv) $W^{(\varepsilon)}$ is a P-a. s. finite and non-degenerate random variable.

Proof.
$(i) \Rightarrow (ii)$ If $\gamma(\varepsilon) > 0$, the ergodic theorem implies that for T defined in (14), Q-a. s.,

$$E_Q\Big(\sum_{k=0}^{T-1} \log R_k\Big) \;=\; E_Q(T)\cdot \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log R_k = E_Q(T)\cdot E_Q(\log R_1) = -\varepsilon^{-1}\cdot\gamma(\varepsilon) < 0.$$

Hence [6, 8] there exists a unique $s > 0$ such that $E_P\big(X_T^{-s}\big) = 1$. It can be shown (see, for instance, display (2.43) in [11]) that this implies $\Lambda_\varepsilon(s) = 0$ and hence $\Lambda_\varepsilon'(0) < 0$.
$(ii) \Rightarrow (i)$ Jensen's inequality implies that $\gamma_\varepsilon \geq -t\Lambda_\varepsilon(t)$, and hence $\gamma(\varepsilon) > 0$ if $\Lambda'(0) < 0$.
$(ii) \Leftrightarrow (iii)$ For any $t > 0$, we have

$$E_P(X_n^{-t}) \geq E_P\Big(X_n^{-t}\prod_{k=2}^{n}(1-\eta_k)\Big) \geq \varepsilon^{n-1} a^{-t(n-1)},$$

and hence $\lim_{t\to\infty} \Lambda_\varepsilon(t) = +\infty$. Since $\Lambda_\varepsilon(t)$ is a convex function with $\Lambda_\varepsilon(0) = 0$, this proves the implication $(ii) \Leftrightarrow (iii)$ (for an illustration, see Fig. 1 below).
$(i) \Leftrightarrow (iv)$ This is the content of Lemma 5.
The proof of the proposition is complete. □

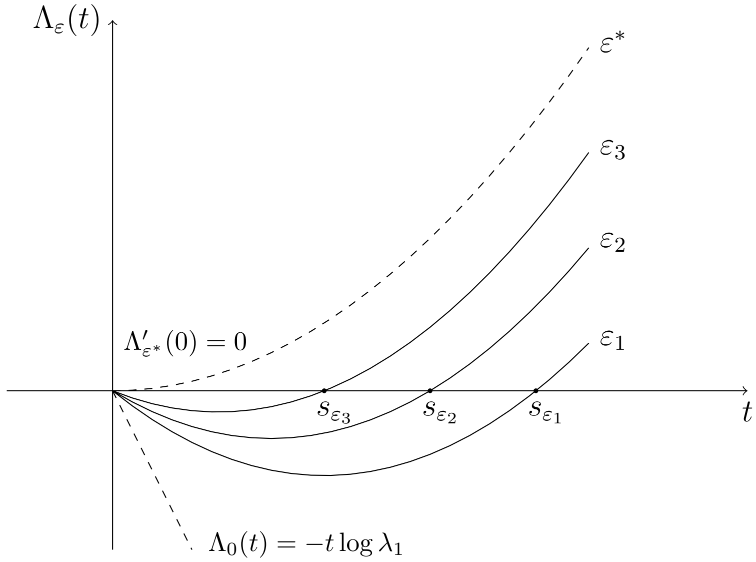

Figure 1: Sketch of the graph of the convex function $\Lambda_\varepsilon(t)$ for 3 increasing parameter values $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$ within the range $(0, \varepsilon^*)$ and the extremal parameter values $\varepsilon = 0$, $\varepsilon = \varepsilon^*$.

Using the notation introduced in the statement of Lemma 3, transition kernel of the time-reversed Markov chain R_{-n} on the state space $\{S_i : i \in \mathbb{N}\}$ can be written as follows:

$$\begin{aligned} H(i,j) &:= Q(R_n = S_j | R_{n+1} = S_i) = \frac{Q(R_{n+1} = S_i | R_n = S_j) Q(R_n = S_j)}{Q(R_n = S_i)} \\ &= \begin{cases} \varepsilon(1-\varepsilon)^{j-1} & \text{if } i = 1 \\ 1 & \text{if } i = j+1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Unfortunately, the infinite matrix $H(i,j)$ doesn't satisfy the conditions imposed in [11, 13] or [3]. More precisely, the kernel H doesn't satisfy the following strong Doeblin condition: $H^m(i,j) \geq c\mu(j)$ for some $m \in \mathbb{N}$, $c > 0$, a probability measure μ on $\mathbb{N}$, and all $i, j \in \mathbb{N}$. However, one can exploit the fact that transition kernel of the Markov chain R_n does satisfy Doeblin's condition with $m = 1$, $c = \varepsilon$, and $\mu = \delta_1$, the degenerate distribution concentrated on $j = 1$. The proof of the following lemma is a mixture of arguments borrowed from [11] and [13]. The key technical ingredient of the proof is the observation that transition kernel of the forward Markov chain R_n satisfies Assumption 1.2 in [13].

Lemma 7. *The claim in part (a)-(iii) of Theorem 1 holds with ε^* introduced in* (17).

Proof. Let $N_0 = 0$ and then for $i \in \mathbb{N}$,

$$N_i = \sup\{k < N_{i-1} : R_{-k} = 1/a\}.$$

Note that the blocks $(R_{N_{i+1}+1}, \ldots, R_{N_i})$ are independent and identically distributed for $i \geq 0$. For $i \geq 0$, let

$$\begin{aligned} A_i &= R_{N_i} + R_{N_i} R_{N_i - 1} + \ldots + R_{N_i} R_{N_i - 1} \ldots R_{N_{i+1}+2} R_{N_{i+1}+1} \\ B_i &= R_{N_i} R_{N_i - 1} \ldots R_{N_{i+1}+1}. \end{aligned}$$

The pairs (A_i, B_i), $i \geq 0$, are independent and identically distributed under the law P. Moreover, it follows from (6) that

$$W^{(\varepsilon)} = A_0 + \sum_{n=1}^{\infty} A_n \prod_{i=0}^{n-1} B_i.$$

To prove Lemma 7 we will verify the conditions of the following Kesten's theorem for $(A_i, B_i)_{i\geq 0}$ under the law P. To enable a further reference (see Section 5 below) we quote this theorem in a more general setting (with not necessarily strictly positive coefficients A_n, B_n) than we actually need for the purpose of proving Lemma 7.

Theorem 8. *[6, 8] Let $(A_i, B_i)_{i\geq 0}$ be i.i.d. pairs of real-valued random variables such that*

(i) For some $s > 0$, $E\big(|A_0|^s\big) = 1$ and $E\big(|B_0|^s \log^+ |B_0|\big) < \infty$, where $\log^+ x := \max\{\log x, 0\}$.

(ii) $P(\log |B_0| = \delta \cdot k$ for some $k \in \mathbb{Z} | B_0 \neq 0) < 1$ for all $\delta > 0$.
Let $W = A_0 + \sum_{n=1}^{\infty} A_n \prod_{i=1}^{n-1} B_i$. Then

(a) $\lim_{t\to\infty} t^s P\big(W > t\big) = K_+$, $\lim_{t\to\infty} t^s P\big(W < -t\big) = K_-$ for some $K_+, K_- \geq 0$.

(b) If $P(B_1 < 0) > 0$, then $K_+ = K_-$.

(c) $K_+ + K_- > 0$ if and only if $P\big(A_0 = (1 - B_0)c\big) < 1$ for all $c \in \mathbb{R}$.

Recall T from (14). Observe that $\log B_0 = \sum_{k=N_1+1}^{0} \log R_k$ is distributed the same as $\sum_{k=1}^{T} \log R_k = \log Z_1 - \log Z_{T+1}$. Therefore the non-lattice condition *(ii)* of the above theorem holds in virtue of (9). Furthermore, since clearly $P(B_0 > 1) > 0$ and $P(B_0 < 1) > 0$, we have $P\big(A_0 = (1 - B_0)c\big) < 1$ for all $c \in \mathbb{R}$.

It remains to verify condition *(i)* of the theorem. Recall S_k introduced in the statement of Lemma 3. Let

$$\widetilde{H}(i,j) \;:=\; Q(R_{n+1} = S_j | R_n = S_i) = \begin{cases} \varepsilon & \text{if } j = 1 \\ 1-\varepsilon & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

be transition kernel of the stationary Markov chain R_n. Between two successive regeneration times N_i the forward chain R_n evolves according to a sub-Markov kernel Θ given by the equation

$$\widetilde{H}(i,j) = \Theta(i,j) + \varepsilon \mathbf{1}(j = 1). \tag{19}$$

That is, for $i, j \in \mathbb{N}$,

$$\Theta(i,j) = Q(R_1 = j, N_1 > 1 | R_0 = i) = \begin{cases} 1-\varepsilon & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Further, for any real $s \geq 0$ define the kernels (countable matrices) $\widetilde{H}_s(i,j)$ and $\Theta_s(i,j)$, $i, j \in \mathbb{N}$, by setting $\widetilde{H}_s(i,j) = \widetilde{H}(i,j)R_j^t$ and $\Theta_s(i,j) = \Theta(i,j)R_j^t$. For an infinite matrix A on $\mathbb{N}$ and a function $f : \mathbb{N} \to \mathbb{R}$ let Af denote the real-valued function on $\mathbb{N}$ with $(Af)(i) := \sum_{j\in\mathbb{N}} A(i,j)f(j)$. Since the forward transition kernel $\widetilde{H}$ satisfies Assumption 1.2 in [13], it follows that from Proposition 2.4 in [13] that for all $s \geq 0$:

1. There exist a real number $\alpha_s > 0$ and a bounded function $f_s : \mathbb{N} \to \mathbb{R}$ such that $\inf_{i\in\mathbb{N}} f_s(i) > 0$ and $\widetilde{H}_s f = \alpha_s f_s$.

2. There exist a real number $\beta_s > 0$ and a bounded function $g_s : \mathbb{N} \to \mathbb{R}$ such that $\inf_{i\in\mathbb{N}} g_s(i) > 0$ and $\Theta_s f = \beta_s f_s$.

3. $\beta_s \in (0, \alpha_s)$.

Without loss of generality we can use the following normalization for the eigenfunctions:

$$f_s(1) = 1. \tag{20}$$

Furthermore, it follows from Lemma 2.3 in [13] that α_s and β_s are spectral norms of infinite matrices $\widetilde{H}_s$ and Θ_s, respectively, and hence are uniquely defined. It follows from Proposition 6 (see Lemma 2.3 in [13]) that $\alpha_{s_\varepsilon} = 1$. In particular, since Λ_ε is a continuous function of s, the spectral radius of Θ_s (regarded as an operator acting on the space of bounded function on $\mathbb{N}$ equipped with the sup-norm) is strictly less than one on an interval $\big(0, \widetilde{s}_\varepsilon\big)$ for some $\widetilde{s}_\varepsilon > s_\varepsilon$. Let I be the infinite unit matrix in $\mathbb{N}$ and $h : \mathbb{N} \to \mathbb{R}$ be a function defined by $h(i) = 1$ for all $i \in \mathbb{N}$. For any $s \in \big(0, \widetilde{s}_\varepsilon\big)$, and in particular for $s = s_\varepsilon$, we have:

$$\begin{aligned} E_P(B_0^s) &= E_P\left(\prod_{k=1}^{T} R_k\right) = E_P\left(\prod_{k=0}^{T-1} R_k\right) = E_P\big[\widetilde{H}_s^T(1,1)\big] \\ &= \sum_{n=1}^{\infty} a^{-s}\varepsilon r \Theta_s^{n-1} h(1) = a^{-s}\varepsilon (I-\Theta_s)^{-1} h(1). \end{aligned}$$

On the other hand, it follows from (19) that for any $i \in \mathbb{N}$ we have

$$f_{s_\varepsilon}(i) = \widetilde{H}_{s_\varepsilon} f_{s_\varepsilon}(i) = \Theta_{s_\varepsilon} f_{s_\varepsilon}(i) + \varepsilon a^{-s_\varepsilon} f_{s_\varepsilon}(1),$$

and hence $E_P(B_0^{s_\varepsilon}) = f_{s_\varepsilon}(1) = 1$, where for the second identity we used (20).

Finally, adapting (2.45) in [11] to our framework we obtain for any $s \in \big(s_\varepsilon, \widetilde{s}_\varepsilon\big)$,

$$\begin{aligned} E_P(A_0^s) &= E_P\left[\left(\sum_{n=1}^{\infty}\sum_{i=1}^{n}\prod_{j=0}^{i-1} R_{-j}\cdot \mathbf{1}(N_1=-n)\right)^s\right] \\ &= \sum_{n=1}^{\infty} E_P\left[\left(\sum_{i=1}^{n}\prod_{j=0}^{i-1} R_{-j}\cdot \mathbf{1}(N_1=-n)\right)^s\right] \\ &\le \sum_{n=1}^{\infty} n^s \sum_{i=1}^{n} E_P\left[\prod_{j=0}^{i-1} R_{-j}^s\cdot \mathbf{1}(N_1=-n)\right] \\ &= \sum_{n=1}^{\infty} n^s \sum_{i=1}^{n} \frac{1}{Q(R_0=1/a)}\cdot E_Q\left[\prod_{j=0}^{i-1} R_{-j}^s\cdot \mathbf{1}(N_1=-n, R_0=1/a)\right] \\ &= \sum_{n=1}^{\infty} n^s \sum_{i=1}^{n} E_P\left[\prod_{j=n-(i-1)}^{n} R_j^s\cdot \mathbf{1}(T=n)\right] = \varepsilon a^{-s}\sum_{n=1}^{\infty} n^s \sum_{i=1}^{n} \Theta^{n-i}\Theta_s^{i-1} h(1) < \infty, \end{aligned}$$

where the last inequality is an implication of the fact that the spectral radius of the infinite positive matrix Θ_s is strictly less than one. The proof of the lemma is complete. $\square$

Remark 9. *The initial conditions $X_0 = 1$ and $X_1 = a$ guarantee that the measure P is Q conditioned on the event $R_0 = 1/a$. If R_0 has a different value, then A_0 and B_0 defined above are still independent of the i. i. d. sequence of pairs $(A_n, B_n)_{n\in\mathbb{N}}$, but the distributions of the pairs (A_0, B_0) and (A_1, B_1) differ in general. Using a slightly more elaborated version of the arguments used in the proof of Lemma 7 (cf. proof of Proposition 2.38 under assumption (1.6) in [11]) it can be shown that all the conclusions of Theorem 1 remain valid for different strictly positive initial values (X_0, X_1) and that the only effect of changing initial conditions is on the value of the constant K_ε.*

To conclude the proof of Theorem 1 it remains to prove the claim in part *(b)* of the theorem. Using the representation of $\Lambda_\varepsilon(t)$ given in (18) and a variation of the coupling argument which we employed in order to prove Proposition 4, we first derive the following auxiliary result:

Lemma 10. *For any fixed $t > 0$, $\Lambda_\varepsilon(t)$ is a strictly increasing function of ε on $[0, 1]$.*

Proof. Recall the notation introduced in the course of the proof of Proposition 4. It follows from the inequality in (16) that

$$\frac{1}{X_n^{(1)}} \;\geq\; \frac{1}{X_n}\cdot\prod_{k=0}^{n-2}\Big(1+\frac{b}{a^2+b}\Big)^{\zeta_k} = \frac{1}{X_n}\cdot\exp\left\{\sum_{k=0}^{n-2}\zeta_k\cdot\log\Big(1+\frac{b}{a^2+b}\Big)\right\}.$$

It follows then from Hölder's inequality that for any constants $t > 0$, $p > 1$ and $q > 0$ such that $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$E_P\left(\frac{1}{X_n^t}\right) \;\leq\; \left[E_P\left(\frac{1}{\big(X_n^{(1)}\big)^{pt}}\right)\right]^{1/p}\cdot\left[E_P\Big(\exp\Big\{-qt\sum_{k=0}^{n-2}\zeta_k\cdot\log\Big(1+\frac{b}{a^2+b}\Big)\Big\}\Big)\right]^{1/q}.$$

Therefore, since ζ_k are i. i. d. Bernoulli random variables,

$$\begin{aligned}
\Lambda_\varepsilon(t) \;&\leq\; \frac{1}{p}\cdot\Lambda_{\varepsilon_1}(pt) + \limsup_{n\to\infty}\frac{1}{nq}\log E_P\Big(\exp\Big\{-qt\sum_{k=0}^{n-2}\zeta_k\cdot\log\Big(1+\frac{b}{a^2+b}\Big)\Big\}\Big)\\
&=\; \frac{1}{p}\cdot\Lambda_{\varepsilon_1}(pt) + \frac{1}{q}\log E_P\Big(\exp\Big\{-qt\zeta_1\cdot\log\Big(1+\frac{b}{a^2+b}\Big)\Big\}\Big)\\
&=\; \frac{1}{p}\cdot\Lambda_{\varepsilon_1}(pt) + \frac{1}{q}\log\Big[(\varepsilon_1-\varepsilon)\cdot\exp\Big\{-qt\log\Big(1+\frac{b}{a^2+b}\Big)\Big\} + (1-\varepsilon_1+\varepsilon)\cdot 1\Big]\\
&=\; \frac{1}{p}\cdot\Lambda_{\varepsilon_1}(pt) + \frac{1}{q}\log\Big[(\varepsilon_1-\varepsilon)\cdot\Big(1+\frac{b}{a^2+b}\Big)^{-qt} + (1-\varepsilon_1+\varepsilon)\Big]\\
&\leq\; \frac{1}{p}\cdot\Lambda_{\varepsilon_1}(pt) - \frac{1}{q}(\varepsilon_1-\varepsilon)\Big[1-\Big(1+\frac{b}{a^2+b}\Big)^{-qt}\Big],
\end{aligned}$$

where in the last step we used the inequality $\log(1-x) < x$. Since $\Lambda_\varepsilon(t)$ is a continuous function of t, by letting p to approach one and thus q to approach infinity, we obtain that

$$\Lambda_{\varepsilon_1}(t) - \Lambda_\varepsilon(t) \geq t(\varepsilon_1-\varepsilon)\log\Big(1+\frac{b}{a^2+b}\Big) > 0.$$

The proof of the lemma is complete. □

We now turn to the proof of part *(b)* of Theorem 1.

Proposition 11. *The critical exponent s_ε is a strictly decreasing continuous function of ε on $[0,\varepsilon^*)$. Furthermore, (4) holds true.*

Proof. The desired monotonicity of s_ε follows directly from Lemma 10, see Fig. 1 above. We will next show that s_ε is a continuous function of ε on $(0,1)$. Due to the monotonicity of s_ε, the following one-sided limits exist for any $\varepsilon \in (0,\varepsilon^*)$:

$$s_\varepsilon^+ = \lim_{\delta\downarrow\varepsilon} s_\delta \qquad \text{and} \qquad s_\varepsilon^- = \lim_{\delta\uparrow\varepsilon} s_\delta.$$

The second limit, namely $s_{\varepsilon^*}^-$, exists also for $\varepsilon=\varepsilon^*$. Set $s_{\varepsilon^*}:=0$. If either $s_\varepsilon^+ > s_\varepsilon$ or $s_\varepsilon^- < s_\varepsilon$ for some $\varepsilon \in [0,\varepsilon^*]$, then (see Fig. 1 above) $\Lambda_\delta(t^*)$ is not a continuous function of δ at any point t^* within the open interval $(s_\varepsilon, s_\varepsilon^+)$ or, respectively, $(s_\varepsilon^-, s_\varepsilon)$. To verify the continuity of s_ε on $(0,\varepsilon^*]$ it therefore suffices to show that $\Lambda_\varepsilon(t)$ is a continuous function of ε for any fixed $t>0$.

We will use again the notation and the coupling construction introduced in the course of the proof of Proposition 4. Recall T_n from (11), and let $T_n^{(1)}$, $n\in\mathbb{N}$, be the corresponding stopping times associated with the sequence $X_n^{(1)}$. Let $\chi_n = \mathbf{1}(T_n \neq T_n^{(1)})$. The random variables χ_n form a two-state Markov chain with transition kernel determined by

$$P(\chi_{n+1}=1|\chi_n=0) = P\big(\eta_n=\eta_n^{(1)}=0\big) = P(\zeta_{n+1}=0) = \varepsilon_1-\varepsilon$$

and

$$P(\chi_{n+1}=1|\chi_n=1) = 1 - P\big(\eta_n=\eta_n^{(1)}=0\big) = 1-\varepsilon.$$

The stationary distribution $\pi = \big(\pi(0),\pi(1)\big)$ of this Markov chain is given by

$$\pi(0) = \frac{\varepsilon}{\varepsilon_1} \qquad \text{and} \qquad \pi(1) = \frac{\varepsilon_1-\varepsilon}{\varepsilon_1}$$

Similarly to (16), in virtue of Lemma 15 we have:

$$\frac{1}{X_n^{(1)}} \;\le\; \frac{1}{X_n}\cdot\prod_{k=0}^{n-2}\Big(\frac{a^2+b}{a}\cdot\frac{1}{a}\Big)^{\xi_k} = \frac{1}{X_n}\cdot\exp\left\{\sum_{k=0}^{n-2}\chi_k\cdot\log\Big(1+\frac{b}{a^2}\Big)\right\}.$$

It follows then from Hölder's and Jensen's inequalities that for any constants $t>0$, $p>1$ and $q>0$ such that $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}
E_P\left(\frac{1}{\big(X_n^{(1)}\big)^t}\right) &\le \left[E_P\left(\frac{1}{X_n^{pt}}\right)\right]^{1/p}\cdot\left[E_P\Big(\exp\Big\{qt\sum_{k=0}^{n-2}\chi_k\cdot\log\Big(1+\frac{b}{a^2}\Big)\Big\}\Big)\right]^{1/q}\\
&\le \left[E_P\left(\frac{1}{X_n^{pt}}\right)\right]^{1/p}\cdot\left[\exp\Big\{qt\sum_{k=0}^{n-2}E_P(\chi_k)\cdot\log\Big(1+\frac{b}{a^2}\Big)\Big\}\right]^{1/q}.\\
&= \left[E_P\left(\frac{1}{X_n^{pt}}\right)\right]^{1/p}\cdot\Big(1+\frac{b}{a^2}\Big)^{t\sum_{k=0}^{n-2}E_P(\chi_k)}.
\end{aligned}$$

Since Markov chain χ_n is aperiodic, its stationary distribution π is the limiting distribution. Thus,

$$\begin{aligned}\Lambda_{\varepsilon_1}(t) &\leq \frac{1}{p}\cdot\Lambda_\varepsilon(pt)+t\log\Big(1+\frac{b}{a^2}\Big)\lim_{n\to\infty}P(\xi_k=1)\\ &= \frac{1}{p}\cdot\Lambda_\varepsilon(pt)+t\pi(1)\cdot\log\Big(1+\frac{b}{a^2}\Big)=\frac{1}{p}\cdot\Lambda_\varepsilon(pt)+t\frac{\varepsilon_1-\varepsilon}{\varepsilon_1}\log\Big(1+\frac{b}{a^2}\Big).\end{aligned}$$

Since $\Lambda_\varepsilon(t)$ is a continuous function of t and $p>1$ is arbitrary, we conclude that

$$0<\Lambda_{\varepsilon_1}(t)-\Lambda_\varepsilon(t)\leq t\frac{\varepsilon_1-\varepsilon}{\varepsilon_1}\log\Big(1+\frac{b}{a^2}\Big),$$

and thus, for a given $t>0$, $\Lambda_\varepsilon(t)$ is a Lipschitz function of the parameter ε on any interval bounded away from zero. This completes the proof of the continuity of s_ε on $(0,\varepsilon^*]$. In particular, the second limit in (4) holds true.

To complete the proof of the proposition it remains to prove that the first limit in (4) holds true, namely $\lim_{\varepsilon\downarrow 0}s_\varepsilon=\infty$. To this end it suffices to show that $s_\varepsilon>t$ for all $\varepsilon>0$ small enough. To this end, observe that since (9) implies $\lim_{n\to\infty}S_n=\lambda_1^{-1}<1$, there exists $k_0\in\mathbb{N}$ such that $S_k<\frac{1}{2}\big(1+\lambda_1^{-1}\big)<1$ for all $k>k_0$. For $n\in\mathbb{N}$, let $\delta_n=\mathbf{1}\big(R_n=S_k \text{ with } k>k_0\big)$ and let $\mathcal{G}_n=\sigma(R_1,R_2,\ldots,R_n)$ be the σ-algebra generated by the random variables R_i with $1\leq i\leq n$. Then, with probability one, we have for $n\geq 2$,

$$\begin{aligned}P\big(\delta_n=0\big|\mathcal{G}_{n-1}\big) &\leq P\Big(\bigcup_{0\leq k\leq k_0}\{\eta_{n-k-2}=0\}\Big|\mathcal{G}_{n-1}\Big)\\ &\leq \sum_{k=0}^{k_0}P\big(\eta_{n-k-2}=0\big|\mathcal{G}_{n-1}\big)=(k_0+1)\varepsilon.\end{aligned}\tag{21}$$

Denote $u=\frac{1}{2}\big(1+\lambda_1^{-1}\big)$ and $v=a^{-1}$. It follows from (18), (21), and (10) that for $\varepsilon<(1+k_0)^{-1}$ we have:

$$\begin{aligned}\Lambda_\varepsilon(t) &= \lim_{n\to\infty}\frac{1}{n}\log E_Q\Big(\prod_{i=1}^n R_i^t\Big)\leq\limsup_{n\to\infty}\frac{1}{n}\log E_Q\Big(\prod_{i=1}^n u^{t\sigma_i}v^{t(1-\sigma_i)}\Big)\\ &\leq \log\big[u^t\big(1-\varepsilon(k_0+1)\big)+v^t\varepsilon(k_0+1)\big].\end{aligned}$$

It thus holds that $\Lambda_\varepsilon(t)<0$, and hence $s_\varepsilon>t$, for all $\varepsilon>0$ small enough. Since $t>0$ is arbitrary, it follows that $\lim_{\varepsilon\downarrow 0}s_\varepsilon=\infty$. The proof of the proposition is complete. □

4 Proof of Proposition 2

For $k\in\mathbb{N}$, et $\mathcal{F}_{k-1}=\sigma(X_0,\eta_0,X_1,\eta_1,\ldots,X_{k-2},\eta_{k-2},X_{k-1},\eta_{k-1},X_k,X_{k+1})$ be the σ-algebra generated by the random variables η_i with $i\leq k-1$ and X_i with $i\leq k+1$. It follows from (1) that η_k is independent of $\mathcal{F}_{k-1}$. In the proof below, we will repeatedly use without further notice the fact $E_P(X\eta_k)=E_P[X(1-\varepsilon)]$ for a random variable $X\in\mathcal{F}_{k-1}$.

Proof.

(a) In order to verify the claim, take the expectation on both sides of (1) and recall (8), (9).

(b) Take the square and then take the expectation on the both sides of (1), to obtain:

$$\begin{aligned}
E_P(X_{n+1}^2) &= a^2 E_P(X_n^2) + b^2(1-\varepsilon)^2 E_P(X_{n-1}^2) + 2b(1-\varepsilon)E_P(aX_nX_{n-1}) \\
&= a^2 E_P(X_n^2) + b^2(1-\varepsilon)^2 E_P(X_{n-1}^2) \\
&\qquad +2b(1-\varepsilon)E_P\big[(X_{n+1} - b\eta_{n-1}X_{n-1})X_{n-1}\big] \\
&= a^2 E_P(X_n^2) - b^2(1-\varepsilon)^2 E_P(X_{n-1}^2) + 2b(1-\varepsilon)E_P(X_{n+1}X_{n-1}) \\
&= \big(a^2 + 2b(1-\varepsilon)\big)E_P(X_n^2) - b^2(1-\varepsilon)^2 E_P(X_{n-1}^2) \\
&\qquad +2b(1-\varepsilon)E_P(h_n),
\end{aligned} \tag{22}$$

where $h_n := X_{n-1}X_{n+1} - X_n^2$.

We will next derive a Cassini-type formula for $E_P(h_n)$. We have:

$$\begin{aligned}
aE_P(h_{n+1}) &= E_P\big[(aX_n)\cdot X_{n+2}\big] - aE_P\big(X_{n+1}^2\big) \\
&= E_P\big[(X_{n+1} - b\eta_{n-1}X_{n-1})\cdot(aX_{n+1} + b\eta_nX_n) - aX_{n+1}^2\big] \\
&= E_P\big[b\eta_nX_nX_{n+1} - ab\eta_{n-1}X_{n-1}X_{n+1} - b^2\eta_{n-1}\eta_nX_{n-1}X_n\big] \\
&= E_P\big[b(1-\varepsilon)X_nX_{n+1} - ab\eta_{n-1}(X_n^2 + h_n) - b^2(1-\varepsilon)\eta_{n-1}X_{n-1}X_n\big] \\
&= E_P\big[b(1-\varepsilon)X_n(X_{n+1} - aX_n - b\eta_{n-1}X_{n-1}) - ab\eta_{n-1}h_n\big] \\
&= -abE_P(\eta_{n-1}h_n).
\end{aligned}$$

Hence,

$$\begin{aligned}
E_P(X_nX_{n+2} - X_{n+1}^2) &= E_P(h_{n+1}) = -bE_P(\eta_{n-1}h_n) = \ldots \\
&= (-b)^n\varepsilon^{n-1}E_p(\eta_0h_1) = (-1)^nb^{n+1}(1-\varepsilon)^n.
\end{aligned} \tag{23}$$

Using the notation $Y_n = E_P(X_n^2)$ and substituting (23) into (22), we obtain

$$Y_{n+1} = \big[a^2 + 2b(1-\varepsilon)\big]Y_n - b^2(1-\varepsilon)^2Y_{n-1} + 2(-b)^{n+1}(1-\varepsilon)^n,$$

from which the claim in *(b)* follows, taking in account that $Y_0 = 1$ and $Y_1 = a^2$.

(c) For any $k \in \mathbb{N}$, we have:

$$\begin{aligned}
U_{n,k+1} &= E_P(X_nX_{n+k+1}) = aE_P(X_nX_{n+k+1}) + bE_P(\eta_{n+k-1}X_nX_{n+k-1}) \\
&= aU_{n,k} + b(1-\varepsilon)U_{n,k-1}.
\end{aligned}$$

Furthermore, using notations introduced in the course of proving *(b)*,

$$\begin{aligned}
E_P(X_nX_{n+1}) &= \frac{1}{a}E_P(X_nX_{n+2}) - \frac{b(1-\varepsilon)}{a}E_P(X_n^2) \\
&= \frac{1}{a}\big[E_P(h_{n+1}) + Y_{n+1}^2 - b(1-\varepsilon)Y_n^2\big].
\end{aligned}$$

The proof of the proposition is complete. □

5 Concluding remarks

1. We believe that K_ε in the statement of Theorem 1 is decreasing as a function of the parameter ε, but were unable to prove it. Some information about this constant can be derived from the formulas given in [2, 6] (see also references in [2]) using the recursion representation (6) of W_n and the regeneration structure described in Section 3 (see the proof of Lemma 7 there) which reduces the Markov setup of this paper to an i. i. d. one considered in [2, 6, 8].

2. We think that s_ε is a strictly convex function of ε on $[0, \varepsilon^*)$, but were unable to prove it. Since $\lim_{\varepsilon\downarrow 0} s_\varepsilon = +\infty$, Fig. 1 strongly suggests that the convexity holds for an interval of small enough values of ε within $(0, \varepsilon^*)$. We believe that, with s_{ε^*} set to zero, s_ε is convex in fact on the whole interval $(0, \varepsilon^*]$.

3. The linear model (1) can serve as an ansatz in a general case. For instance, it seems plausible that a result similar to our Theorem 1 holds for generalized Fibonacci sequences considered in [7]. This is a work in progress by the authors.

4. Using appropriate variations of the above Proposition 6 and Lemma 7, Theorem 1 can be extended to a class of recursions $\widetilde{W}_{n+1} = \theta \cdot \prod_{i=0}^{m-1} R_{nl+i}^{h_i} \widetilde{W}_n + Q_n$ with arbitrary $l, m \in \mathbb{N}$, positive reals h_i, and suitable coefficients θ (large enough by absolute value constant) and Q_n (in general random). For instance, in the spirit of [9], one can consider sequences $\widetilde{W}_n = \frac{1}{X_{2n}^2} \sum_{k=0}^{n-1} X_{2k+1} X_{2k+8}$ or $\widetilde{W}_n = \frac{1}{X_n^2} \sum_{k=0}^{n-1} (-1)^k X_k^2$. The former case corresponds to $\widetilde{W}_{n+1} = Q_n \widetilde{W}_n + Q_n$ with $Q_n = R_{2n+1} R_{2n+2} R_{2n+8} R_{2n+9}$, and the later to $\widetilde{W}_{n+1} = Q_n \widetilde{W}_n + 1$ with $Q_n = (-1)^n R_n^2$. We leave details to the reader.

References

[1] E. Ben-Naim and P. L. Krapivsky, *Weak disorder in Fibonacci sequences*, J. Phys. A **39** (2006), L301–L307.

[2] D. Buraczewski, E. Damek, and J. Zienkiewicz, *On the Kesten-Goldie constant*, J. Difference Equ. Appl. **22** (2016), 1646–1662.

[3] J. F. Collamore, *Random recurrence equations and ruin in a Markov-dependent stochastic economic environment*, Ann. Appl. Probab. **19** (2009), 1404–1458.

[4] R. Durrett, *Probability: Theory and Examples*, 4th ed., Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2010.

[5] H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Statist. **31** (1960), 269–556.

[6] C. M. Goldie, *Implicit renewal theory and tails of solutions of random equations*, Ann. Appl. Probab. **1** (1991), 126–166.

[7] É. Janvresse, B. Rittaud, and T. de la Rue, *Almost-sure growth rate of generalized random Fibonacci sequences*, Ann. Inst. H. Poincaré Probab. Statist. **46** (2010), 1–298.

[8] H. Kesten, *Random difference equations and renewal theory for products of random matrices*, Acta. Math. **131** (1973), 208–248.

[9] T. Koshy, *Fibonacci and Lucas Numbers with Applications* (Pure and Applied Mathematics: A Wiley-Interscience Series of Texts, Monographs and Tracts), Wiley-Interscience, 2001.

[10] Y. Lan, *Novel computation of the growth rate of generalized random Fibonacci sequences*, J. Stat. Phys. **142** (2011), 847–861.

[11] E. Mayer-Wolf, A. Roitershtein, and O. Zeitouni, *Limit theorems for one-dimensional transient random walks in Markov environments*, Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), 635–659.

[12] Y. Peres, *Domains of analytic continuation for the top Lyapunov exponent*, Ann. Inst. H. Poincaré Probab. Statist. **28** (1992), 131–148.

[13] A. Roitershtein, *One-dimensional linear recursions with Markov-dependent coefficients*, Ann. Appl. Probab. **17** (2007), 572–608.

[14] O. Zeitouni, *Random Walks in Random Environment*, XXXI Summer School in Probability, (St. Flour, 2001). Lecture Notes in Math. **1837**, Springer, 2004, pp. 193-312.

[15] C. Zhang and Y. Lan, *Computation of growth rates of random sequences with multi-step memory*, J. Stat. Phys. **150** (2013), 722–743.