Limit theorem for the Robin Hood game

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Abstract

In its simplest form, the Robin Hood game is described by the following urn scheme: every day the Sheriff of Nottingham puts s balls in an urn. Then Robin chooses r ($r < s$) balls to remove from the urn. Robin's goal is to remove balls in such a way that none of them are left in the urn indefinitely. Let T_n be the random time that is required for Robin to take out all $s \cdot n$ balls put in the urn during the first n days. Our main result is a limit theorem for T_n if Robin selects the balls uniformly at random. Namely, we show that the random variable $T_n \cdot n^{-s/r}$ converges in law to a Fréchet distribution as n goes to infinity.

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1 Introduction and statement of main results

In this paper we describe a limit theorem for the extreme order statistics of triangular arrays of certain dependent random variables. The result can be naturally formulated in terms of an urn scheme which — following [4] — we call the *probabilistic Robin Hood game*.

Fix some parameters $s, r \in \mathbb{N}$ with $r < s$. Imagine that every day the Sheriff of Nottingham puts s balls (bags of gold) in an urn (a mystery cave), after which Robin Hood removes r balls chosen at random from the urn. Let T_n be the time that is required for Robin to take out all $s \cdot n$ balls put in the urn during the first n days. Let

$$w = s - r, \tag{1}$$

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and enumerate arbitrarily the $w \cdot n$ balls found in the urn at the end of the n-th day. We have

$$T_n = n + \max_{1 \le i \le wn} \tau_i^{(n)}, \tag{2}$$

where $\tau_i^{(n)}$ is the time required for Robin to take out the i-th ball. We remark that this urn scheme itself does not describe any strategic game, rather it provides a stochastic version of a play of a certain two-person game previously studied by logicians; see [4] for more details.

Theorem 1. *Let $w = s - r$. We have that for any $x > 0$,*

$$\lim_{n\to\infty} \mathbb{P}(T_n > n + xn^{s/r}) = 1 - \exp\bigl(-wx^{-r/w}\bigr), \tag{3}$$

Theorem 1 follows from a more general result stated in Theorem 2 below. The limiting distribution function $F(x) = \exp\bigl(-wx^{-r/w}\bigr)$ belongs to the Fréchet family of extreme value distributions (see for instance [2, 11]). Let $a_n \sim b_n$ denote $\lim_{n\to\infty} a_n/b_n = 1$. It follows from (12) below that, for a fixed $x > 0$, we have $\mathbb{P}\bigl(\tau_i^{(n)} > xn^{s/r}\bigr) \sim x^{-r/w} \cdot n^{-1}$ as n goes to infinity. Hence

$$\mathbb{P}\bigl(T_n > n + xn^{s/r}\bigr) = \mathbb{P}\bigl(\max_{1\le i\le wn} \tau_i^{(n)} > xn^{s/r}\bigr) \sim 1 - \bigl\{\mathbb{P}\bigl(\tau_i^{(n)} \le xn^{s/r}\bigr)\bigr\}^{wn}. \tag{4}$$

In other words, even though the random variables $\tau_i^{(n)}$ are not independent, the distribution tail of $T_n - n = \max_{1\le i\le wn} \tau_i^{(n)}$ is asymptotically close to that of the maximum of $w \cdot n$ independent random variables, each distributed as $\tau_i^{(n)}$.

The asymptotic relation (4) indicates that the correlation between $\tau_i^{(n)}$ and $\tau_j^{(n)}$ for fixed $i, j \in \mathbb{N}$ becomes weak when n is getting large. There is a large literature on the extreme value theory for sequences of weakly dependent random variables (see for instance [1, 5, 6, 7, 8, 10, 12] and references therein). Interesting applications of such results, for instance to meteorology and actuarial science, can be found e.g. in [2, 9, 14, 15].

Since $T_n - n$ is a waiting time until a certain set of balls is collected by Robin, the study of the asymptotic behavior of T_n can be thought of as a twist on the coupon collector problem, cf. [3] and [1, 5, 10]. The proof of (3) given in Section 2 is combinatorial, and it is based on the asymptotic analysis of an inclusion-exclusion formula (see (8) below) for $\mathbb{P}(T_n > n + xn^{s/r})$.

Several deterministic variations of the Robin Hood game have been studied by logicians and computer scientists, the probabilistic version and its interpretation in terms of the characters of the Robin Hood story were introduced in [4].

Without loss of generality we can assume that the bags in the cave are labeled by integer numbers, preserving the weak order induced by the date of arrival, so that in day n the new bags are labeled by $sn - s + 1, \ldots, sn$. Let D_n be the set of bags that Robin finds in the cave at night n, and let $W_n \subset D_n$ denote the set of the bags removed from the cave at night n. Thus

$$D_n = \{1, \ldots, sn\} \setminus \cup_{i=1}^{n-1} W_i. \tag{5}$$

For a finite set A let $|A|$ denote its size. With $w = s - r$, note that the total number of bags at the cave just before Robin's visit at night n is $|D_n| = w(n-1) + s = wn + r$.

Formally, the *randomized strategy of Robin* is a probability law $\mathbb{P}$ of a random sequence $(W_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$,

$$\mathbb{P}(W_n = A|W_1, \dots, W_{n-1}) = \begin{cases} \binom{wn+r}{r}^{-1} & \text{if } A \subset D_n \text{ and } |A| = r \\ 0 & \text{otherwise,} \end{cases}$$

where D_n is given in terms of the W_i's in (5). Since any ball in the urn at day n has probability $\frac{r}{wn+r}$ of being removed, it follows from the Borel-Cantelli lemma that with probability one, every ball will be eventually removed from the urn by Robin (see [4, 13]). The random time T_n introduced in (2) can be alternatively defined as

$$T_n = \min\{m > n : D_n \cap D_m = \emptyset\}.$$

The following theorem is the main result of this paper.

Theorem 2. *With the above notations,*

(a) For any $x > 0$ and $\alpha < 1$ we have,

$$\left|\mathbb{P}(T_n \le n + xn^{s/r}) - \exp\bigl(-wx^{-r/w}\bigr)\right| = o(n^{-\alpha}). \tag{6}$$

In particular, (3) *holds.*

(b) Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive reals such that $\lim_{n\to\infty} n^{-s}t_n^r = \infty$. Then

$$\mathbb{P}(T_n > n + t_n) \sim wn^{s/w}t_n^{-r/w}.$$

(c) For any fixed $n \in \mathbb{N}$, there exists $K_n \in (0, \infty)$ such that

$$\lim_{x\to\infty} x^{r/w}\mathbb{P}(T_n > n + x) = K_n \tag{7}$$

Moreover, as $n \to \infty$ we have $K_n \sim wn^{s/w}$.

Part (a) of the theorem is just Theorem 1 with additional control over the rate of convergence. The limit theorem stated in Theorem 1 can be rewritten in the following form

$$\mathbb{P}(T_n > n + t_n) \sim 1 - \bigl(\mathbb{P}(\tau_n \le t_n)\bigr)^{wn} \sim 1 - \exp\bigl\{-wn \cdot (n/t_n)^{r/w}\bigr\},$$

with $t_n = xn^{s/r}$ for some fixed x. Part (b) of Theorem 2 shows that this asymptotic relation holds in fact for a wide range of sequences t_n. Finally, part (c) contains a variant of the previously stated asymptotic results for a given value of n by a change in the order of the limits.

The proof of Theorem 2 is included in Section 2.

2 Proof of Theorem 2

First, we obtain an approximation for $\mathbb{P}(T_n > n+t)$ via the inclusion-exclusion formula. Recall that D_n is the set of bags in the cave just before Robin's nth move. For any $n, t \in \mathbb{N}$ and $\mathcal{S}$, define the event

$$A^{n,t}_{\mathcal{S}} = \{\mathcal{S} \subset D_n \cap D_{n+t}\},$$

and let

$$B^{n,t}_{\mathcal{S}} = \mathbb{P}(A^{n,t}_{\mathcal{S}} | \mathcal{S} \subset D_n).$$

By symmetry, $B_{\mathcal{S}}$ depends only on $|S|$. Thus for a set of size m we simplify the notation to $B^{n,t}_m$. By the definition of T_n, we have

$$\{T_n > n+t\} = \bigcup_{i \in D_n} A^{n,t}_{\{i\}}.$$

It follows from the inclusion-exclusion formula that for any integer $L \le wn + r$,

$$(-1)^{L-1}\Big\{\mathbb{P}(T_n > n+t) - \sum_{m=1}^{L} (-1)^{m-1} \binom{wn+r}{m} B^{n,t}_m\Big\} \le 0. \tag{8}$$

In what follows (see (14) below) we show that, for a suitable choice of t_n,

$$\lim_{n\to\infty} \mathbb{P}(T_n > n+t_n) = \lim_{L\to\infty} \lim_{n\to\infty} \sum_{m=1}^{L} (-1)^{m-1} \binom{wn+r}{m} B^{n,t_n}_m.$$

The proof of of our main results relies on this observation.

We first outline the argument leading to (3), and then give a detailed proof of (6), from which (3) follows. We start with the estimation of the probabilities $B^{n,t}_m$. Recall that $|D_n| = wn + r$. We have:

$$\begin{aligned} B^{n,t}_m &= \prod_{i=1}^{t} \frac{\binom{w(n+i)+r-m}{r}}{\binom{w(n+i)+r}{r}} = \prod_{i=1}^{t} \frac{(w(n+i))!(w(n+i)-m+r)!}{(w(n+i)-m)!(w(n+i)+r)!} \\ &= \prod_{i=1}^{t}\prod_{k=1}^{m} \frac{w(n+i)-m+k}{w(n+i)+r-m+k} = \prod_{k=1}^{m}\prod_{i=1}^{t}\Big(1 - \frac{r}{w(n+i)+r-m+k}\Big). \end{aligned} \tag{9}$$

Therefore, the following asymptotic relation can be justified for suitable sequences $(t_n)_{n\in\mathbb{N}}$ (we specify our choice in (11) below). For a fixed $m \in \mathbb{N}$, as $n \to \infty$,

$$B^{n,t_n}_m \sim \prod_{k=1}^{m} \exp\Big\{ - \sum_{i=1}^{t_n} \frac{r}{w(n+i)+r}\Big\} \sim \exp\Big\{-m \int_1^{t_n} \frac{r}{w(n+t)+r}\,dt\Big\}. \tag{10}$$

Fix any $x > 0$ and let

$$t_n = [xn^{s/r}], \tag{11}$$

where $[y]$ denotes the integer part of $y \in \mathbb{R}$, that is $[y] = \max\{n \in \mathbb{Z} : n \leq y\}$. Letting n go to infinity we obtain from (10) for a fixed $m \in \mathbb{N}$,

$$B_m^{n,[xn^{s/r}]} \sim \exp\Big\{-m \int_1^{xn^{s/r}} \frac{r}{(n+t)w+r} dt\Big\} \sim \big(x^{r/w} n\big)^{-m}. \tag{12}$$

On the other hand, $\binom{wn+r}{m} \sim (wn)^m/m!$ as n goes to infinity and m remains fixed. It follows from (8) that (using odd $L = 2l+1$)

$$\limsup_{n\to\infty} \mathbb{P}(T_n > n + xn^{s/r}) \leq \liminf_{l\to\infty} \sum_{m=1}^{2l+1} (-1)^{m-1} \frac{(wn)^m}{m!} \big(x^{r/w} n\big)^{-m}$$

and (using even $L = 2l$)

$$\liminf_{n\to\infty} \mathbb{P}(T_n > n + xn^{s/r}) \geq \limsup_{l\to\infty} \sum_{m=1}^{2l} (-1)^{m-1} \frac{(wn)^m}{m!} \big(x^{r/w} n\big)^{-m}.$$

Therefore the following limit exists for any $x > 0$:

$$\lim_{n\to\infty} \mathbb{P}(T_n > n + xn^{s/r}) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{(wn)^m}{m!} \big(x^{r/w} n\big)^{-m} = 1 - \exp\big(-wx^{-r/w}\big).$$

This completes the proof of (3), provided that (10) has been established. Using (9), we next obtain an upper and a lower bounds for $B_m^{n,t}$ from which (10) will follow.

Upper bound for $B_m^{t,n}$.

We will, without any further mention, use the following inequalities:

$$1 + x < e^x \qquad \text{and} \qquad 1 - x < e^{-x} < 1 - x + \frac{x^2}{2}$$

for arbitrary $x > 0$,

$$1 - x\alpha < (1-x)^{\alpha}$$

for $x \in (0,1)$ and $\alpha > 0$, and

$$\int_1^{t+1} g(x)dx \leq \sum_{i=1}^{t} g(i) \leq \int_0^t g(x)dx$$

for a positive, non-increasing function $g(x)$, $x > 0$.

Observe that for any $n, t \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $m < wn + r$,

$$\begin{aligned}
\binom{wn+r}{m} \cdot B_m^{t,n} &\le \frac{(wn+r)^m}{m!} \exp\Big\{-\sum_{i=1}^{t} \frac{r}{w(n+i)+r}\Big\} \\
&\le \frac{(wn+r)^m}{m!} \exp\Big\{-m \int_1^{t+1} \frac{r}{w(n+x)+r} dx\Big\} = \frac{(wn+r)^m}{m!} \Big(\frac{w(n+1)+r}{w(n+t)+r}\Big)^{rm/w} \\
&= \frac{1}{m!} \frac{(wn+s)^{sm/w}}{\big(w(n+t)+r\big)^{rm/w}} \le \frac{1}{m!} \frac{(wn+s)^{sm/w}}{(wt+s)^{rm/w}}.
\end{aligned} \tag{13}$$

To derive (6) from (8) we will use the following upper bound for $B_m^{t,n}$.

Lemma 1. *For any $n, t \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $m < wn + r$, we have*

$$\binom{wn+r}{m} \cdot B_m^{t,n} \le \frac{w^m}{m!} \cdot \frac{n^{sm/w}}{t^{rm/w}} + \frac{w^m}{m!} \cdot \frac{n^{sm/w}}{t^{rm/w}} \cdot \big(e^{\frac{ms^2}{nw^2}} - 1\big) = \frac{w^m}{m!} \cdot \frac{n^{sm/w}}{t^{rm/w}} \cdot e^{\frac{ms^2}{nw^2}}.$$

Proof of Lemma 1. We have:

$$\begin{aligned}
0 &\le \frac{(wn+s)^{sm/w}}{(wt+s)^{rm/w}} - w^m \cdot \frac{n^{sm/w}}{t^{rm/w}} \\
&= \Big(\frac{n^s}{t^r}\Big)^{m/w} \cdot w^m \cdot \Big\{\Big(1 + \frac{s}{wn}\Big)^{sm/w} \cdot \Big(1 + \frac{s}{wt}\Big)^{-rm/w} - 1\Big\} \\
&\le \Big(\frac{n^s}{t^r}\Big)^{m/w} \cdot w^m \cdot \Big\{\Big(1 + \frac{s}{wn}\Big)^{sm/w} - 1\Big\} \le \Big(\frac{n^s}{t^r}\Big)^{m/w} \cdot w^m \cdot \big(e^{\frac{ms^2}{nw^2}} - 1\big),
\end{aligned}$$

completing the proof of the lemma in view of (13). □

Lower bound for $B_m^{t,n}$.

We need the following simple result (compare, for instance, with Lemma 4.3 in [3, p. 112]).

Lemma 2. *Let $(a_n)_{n\in\mathbb{Z}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of positive numbers such that $a_n \le b_n$ and $b_{n+1} \le b_n$ for any $n \in \mathbb{N}$. Then*

$$\prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i \le \prod_{i=1}^{n-1} b_i \cdot \sum_{i=1}^{n} (b_i - a_i).$$

Proof of Lemma 2. We have

$$\prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i = b_n \cdot \Big(\prod_{i=1}^{n-1} b_i - \prod_{i=1}^{n-1} a_i\Big) + (b_n - a_n) \cdot \prod_{i=1}^{n-1} a_i.$$

Iterating we obtain:

$$\prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i \le \sum_{i=1}^{n} (b_i - a_i) \cdot \prod_{j=1}^{i-1} a_i \cdot \prod_{j=i+1}^{n} b_j \le \sum_{i=1}^{n} (b_i - a_i) \cdot \prod_{j=1}^{n-1} b_i,$$

where both conditions of the lemma are used in the last step. $\square$

To derive (6) from (8) we will use the following upper bound for $B_m^{t,n}$.

Lemma 3. *For any $n, t \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $m < \sqrt{\frac{w^2 n}{s}}$, we have*

$$\binom{wn+r}{m} \cdot B_m^{t,n} \ge$$
$$\frac{w^m}{m!} \cdot \Big(\frac{n^s}{t^r}\Big)^{m/w} \cdot \Big(1 - \frac{sm^2}{w^2 n}\Big) e^{-\frac{rmn}{wt}} - \frac{w+s}{s} \cdot \frac{w^m}{m!} \cdot \Big(\frac{n^s}{t^r}\Big)^{m/w} \cdot \frac{mr}{wn-m}.$$

Proof of Lemma 3. It follows from Lemma 2 that for any $n, t \in \mathbb{N}$ and $m \in \mathbb{N}$ such that $m < wn$,

$$\begin{aligned}
0 &\le \exp\Big(-\sum_{i=1}^{t} \frac{mr}{(n+i)w - m + r}\Big) - B_m^{t,n} \\
&\le \prod_{i=1}^{t} \exp\Big(-\frac{mr}{(n+i)w - m + r}\Big) - \prod_{i=1}^{t} \Big(1 - \frac{mr}{(n+i)w - m + r}\Big) \\
&\le \frac{1}{2} \exp\Big(-\sum_{i=1}^{t-1} \frac{mr}{(n+i)w + r}\Big) \cdot \sum_{i=1}^{t} \Big(\frac{mr}{(n+i)w - m + r}\Big)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &\le \exp\Big(-\sum_{i=1}^{t} \frac{mr}{(n+i)w - m + r}\Big) - B_m^{t,n} \\
&\le \frac{1}{2} \exp\Big(-\int_1^t \frac{mr}{(n+s)w + r} ds\Big) \cdot \int_0^t \Big(\frac{mr}{(n+s)w - m + r}\Big)^2 ds \\
&\le \frac{1}{2} \Big(\frac{w(n+1) + r}{w(n+t) + r}\Big)^{mr/w} \cdot \frac{mr}{w(nw - m + r)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\exp\Big(-\sum_{i=1}^{t} \frac{mr}{(n+i)w - m + r}\Big) &\ge \exp\Big(-\int_0^t \frac{mr}{(n+s)w - m + r} ds\Big) \\
&= \Big(\frac{nw - m + r}{(n+t)w - m + r}\Big)^{mr/w} \ge \Big(\frac{nw - m}{(n+t)w - m}\Big)^{mr/w},
\end{aligned}$$

and hence

$$\Big(\frac{nw-m}{(n+t)w-m}\Big)^{mr/w}-B_m^{t,n}\leq\frac{1}{2}\Big(\frac{w(n+1)+r}{w(n+t)+r}\Big)^{mr/w}\cdot\frac{mr}{w(nw-m+r)}.$$

It follows that

$$\binom{wn+r}{m}\cdot B_m^{t,n}\geq\Big(\frac{(wn-m)^s}{(wn+wt)^r}\Big)^{m/w}-\Big(\frac{(wn+s)^s}{(wt)^r}\Big)^{m/w}\cdot\frac{mr}{wn-m}.$$

Let $\lambda=\frac{w+s}{s}$. Then $\frac{(wn+s)^s}{(wt)^r}\leq\lambda\frac{(wn)^s}{(wt)^r}$ for any $n\in\mathbb{N}$, and hence

$$\binom{wn+r}{m}\cdot B_m^{t,n}\geq\Big(\frac{(wn-m)^s}{(wn+wt)^r}\Big)^{m/w}-\lambda w^m\Big(\frac{n^s}{t^r}\Big)^{m/w}\cdot\frac{mr}{wn-m}.$$

Finally, for $m<\sqrt{\frac{w^2n}{s}}$,

$$\begin{aligned}\frac{(wn-m)^{sm/w}}{(wt+wn)^{rm/w}}-w^m\cdot\frac{n^{sm/w}}{t^{rm/w}}&=\Big(\frac{n^s}{t^r}\Big)^{m/w}\cdot w^m\cdot\Big\{\Big(1-\frac{m}{wn}\Big)^{sm/w}\cdot\Big(1+\frac{n}{t}\Big)^{-rm/w}-1\Big\}\\ &\geq\Big(\frac{n^s}{t^r}\Big)^{m/w}\cdot w^m\cdot\Big\{\Big(1-\frac{sm^2}{w^2n}\Big)e^{-\frac{rmn}{wt}}-1\Big\},\end{aligned}$$

completing the proof of Lemma 3. □

We are now in a position to complete the proof of Theorem 2.

Completion of the proof of Theorem 2.

Part (a) of Theorem 2. We now turn to the proof of (6). For a fixed $\alpha\in(0,1)$ let

$$\varepsilon=\frac{1}{3}\min\{1-\alpha,w/r-\alpha\}.$$

For $n\in\mathbb{N}$, let $L_{\alpha,n}=[n^\varepsilon]$ in (8). In particular, $L_{\alpha,n}<\sqrt{\frac{w^2n}{s}}$ for all n large enough. It follows from the bounds stated in Lemma 1 and Lemma 3 that for $t_n=[xn^{s/r}]$ with any $x>0$,

$$\begin{aligned}&\Big|\mathbb{P}\big(T_n>n+t_n\big)-\sum_{m=1}^{L_{\alpha,n}}(-1)^{m-1}\frac{1}{m!}\cdot\Big(\frac{wn}{[nx^{r/w}]}\Big)^m\Big|\leq\\ &\quad\exp\big\{(w+1)x^{-r/w}\big\}\\ &\qquad\times\Big\{1-\Big(1-\frac{s\big(L_{\alpha,n}\big)^2}{w^2n}\Big)e^{-\frac{rnL_{\alpha,n}}{wt_n}}+\frac{w+s}{s}\cdot\frac{rL_{\alpha,n}}{wn-L_{\alpha,n}}+\big(e^{\frac{s^2L_{\alpha,n}}{nw^2}}-1\big)\Big\}.\end{aligned}$$

First, observe that

$$\lim_{n\to\infty} n^\alpha \times \Big\{1-\Big(1-\frac{s\big(L_{\alpha,n}\big)^2}{w^2 n}\Big)e^{-\frac{rnL_{\alpha,n}}{wt_n}} + \frac{w+s}{s}\cdot\frac{rL_{\alpha,n}}{wn-L_{\alpha,n}} + \big(e^{\frac{s^2 L_{\alpha,n}}{nw^2}}-1\big)\Big\} = 0,$$

due to the definition of ε and $L_{\alpha,n}$. To conclude the proof of part (a) of the theorem it remains to observe that

$$\begin{aligned}&\Big|\exp\{-wx^{-r/w}\} - \sum_{m=1}^{L_{\alpha,n}}(-1)^{m-1}\frac{1}{m!}\cdot\Big(\frac{wn}{[nx^{r/w}]}\Big)^m\Big|\\ &\qquad\le \frac{1}{L_{\alpha,n}!}\cdot\Big(\frac{wn}{[nx^{r/w}]}\Big)^{L_{\alpha,n}}\cdot\exp\{(w+1)x^{-r/w}\},\end{aligned}$$

and hence

$$\lim_{n\to\infty} n^\alpha \times \Big|\exp\{-wx^{-r/w}\} - \sum_{m=1}^{L_{\alpha,n}}(-1)^{m-1}\frac{1}{m!}\cdot\Big(\frac{wn}{[nx^{r/w}]}\Big)^m\Big| = 0. \tag{14}$$

Part (b) of Theorem 2. For any fixed $L \in \mathbb{N}$, formula (8) gives an approximation for the distribution tail of T_n. For $B_m^{t,n}$ we have

$$B_m^{t,n} \sim \prod_{k=1}^m \exp\Big\{-\int_1^{t_n}\frac{r}{(n+s)w-m+k}ds\Big\} = \big(t_n/n\big)^{-rm/w}, \qquad n\to\infty.$$

Since $t_n/n \to_{n\to\infty} \infty$, the asymptotic behavior of the tail is determined already by the first term in (8), with $m=1$. That is,

$$\lim_{n\to\infty}(t_n^r/n^s)^{1/w}\cdot\mathbb{P}(T_n > n+t_n) = \lim_{n\to\infty}(t_n^r/n^s)^{1/w}\cdot wn\cdot(t_n/n)^{-r/w} = w.$$

The proof of part (b) of the theorem is thus completed.

Part (c) of Theorem 2. By the inclusion-exclusion formula,

$$\mathbb{P}(T_n > n+t) = \sum_{m=1}^{wn+r}(-1)^{m-1}\binom{wn+r}{m}B_m^{t,n}.$$

It follows from Lemma 1 that $\lim_{t\to\infty} t^{r/w}B_m^{t,n} = 0$ for all $m>1$, and hence the first term in this expansion determines the asymptotic form of the distribution tail of $T_n - n$. Thus the following limit exists

$$K_n = \lim_{t\to\infty} t^{r/w}\cdot\mathbb{P}(T_n > n+t) = \lim_{t\to\infty} t^{w/s}\cdot(wn+r)\prod_{i=1}^t\Big(1-\frac{r}{(n+i)w+r}\Big) \in (0,\infty),$$

and, furthermore, as $n\to\infty$,

$$\begin{aligned}K_n &\sim \lim_{t\to\infty} t^{r/w}\cdot(wn+r)\,\exp\Big(-\int_0^t\frac{r}{(n+x)w+r}dx\Big)\\ &= \lim_{t\to\infty} t^{r/w}\cdot(wn+r)\,\Big(\frac{nw+r}{(n+t)w+r}\Big)^{r/w} = \frac{(nw+r)^{s/w}}{w^{r/w}} \sim wn^{s/w}.\end{aligned}$$

The proof of Theorem 2 is completed. $\square$

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