Iterated Routh's triangles

Emily Carroll, Arka P. Ghosh, Xuan Hien Nguyen,
and Alexander Roitershtein

March 11, 2017; Revised September 4, 2017

Abstract

We consider a series of iterated Routh's triangles. In a general deterministic case we find the limit point of the sequence. We discuss a representation of the limit as a fixed point of a 3-dimensional affine transformation and a curious interpretation of the iterative process as a 3-person job allocation procedure. For a random sequence of iterations, we show that the expected value of the limiting point is the centroid of the original triangle.

1 Introduction

It is well known that the medians of a triangle are concurrent. In general, three interior lines out of the vertices of a triangle form a smaller triangle in its interior, called a Routh's triangle. Only under special circumstances do these interior lines intersect at one point, and Ceva's theorem (see Theorem 1 below) provides a necessary and sufficient condition for the concurrence. The aim of this paper is to study the convergence of a general sequence of nested Routh's triangles. Although a part of our results remains valid when some of the lines are exterior, for simplicity we focus on interior lines.

Here and henceforth, a triangle is a closed convex polygon with three distinct vertices in $\mathbb{R}^2$ and $\mathbb{Z}_+$ stands for the set of non-negative integers. Let $\triangle A_0B_0C_0$ be a triangle and $(x_n, y_n, z_n) \in (0,\infty)^3$, $n \in \mathbb{Z}_+$, a given sequence of triplets of positive numbers. To comply with the limitation imposed by Ceva's theorem (see Theorem 1 below) we assume throughout this paper:

$$x_n y_n z_n \neq 1, \qquad \forall\ n \in \mathbb{Z}_+. \tag{1}$$

A Routh's triangle $\triangle A_{n+1}B_{n+1}C_{n+1}$ is constructed within the interior of $\triangle A_nB_nC_n$ as shown in Figure 1 below using the following scheme:

$$\overrightarrow{B_nA_n'} = -x_n\overrightarrow{C_nA_n'}, \qquad \overrightarrow{C_nB_n'} = -y_n\overrightarrow{A_nB_n'}, \qquad \overrightarrow{A_nC_n'} = -z_n\overrightarrow{B_nC_n'}, \tag{2}$$

A_{n+1} is the intersection point of the straight segments B_nB_n' and C_nC_n', B_{n+1} the intersection of C_nC_n' and A_nA_n', and C_{n+1} the intersection of A_nA_n' and B_nB_n'.

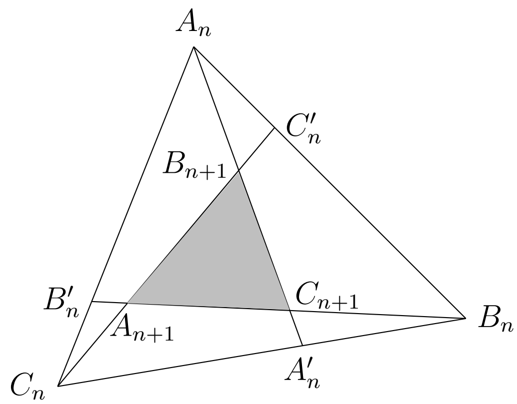

Figure 1: The n-th iteration step in the nested Routh's triangles process. The points A_n' on the triangle side B_nC_n, B_n' on C_nA_n, and C_n' on A_nB_n are chosen according to the rule specified in (2). A_{n+1} is the intersection point of the straight segments B_nB_n' and C_nC_n', B_{n+1} the intersection of C_nC_n' and A_nA_n', and C_{n+1} the intersection of A_nA_n' and B_nB_n'.

We denote the triangle (polygon together with its boundary) $\triangle A_nB_nC_n$ by T_n and its area by Δ_n. We refer to the line segments A_nA_n', B_nB_n' and C_nC_n' as cevians and to the triple (x_n, y_n, z_n) as cevian ratios. Condition (1) ensures that the triangles T_n are non-degenerate.

Theorem 1 (Ceva's theorem)**.** *Assume that T_n is a non-degenerate triangle. Then the cevians A_nA_n', B_nB_n' and C_nC_n' are concurrent if and only if $x_ny_nz_n = 1$.*

Theorem 2 (Routh's theorem)**.** $\Delta_{n+1} = \Delta_n \cdot R(x_n, y_n, z_n)$, *where*

$$R(x,y,z) := \frac{(xyz-1)^2}{(1+x+xy)(1+y+yz)(1+z+xz)}. \tag{3}$$

Ceva's theorem is named after the Italian mathematician Giovanni Ceva, who published this result in 1678. The theorem is closely related to the Menelaus theorem, and was known at least as early as in the eleventh century by Al-Mutaman ibn Hüd, a ruling King of Zaragoza [24]. For discussions on and generalizations of Ceva's theorem see, for instance, [3, 20, 28, 29, 33, 38, 43]. Notably, in [23] Ceva's theorem is applied to analyze a connection between two psychometric models, the Bradley-Terry-Luce model of a pairwise data comparison and the Rasch measurement model. Ceva's concurrence condition is implied by Routh's theorem. For various proofs and extensions of Routh's area formula, we refer the reader to [5, 11, 26, 27, 32, 36], see also references therein. In particular, [32] includes a comprehensive list of references on the topic. We remark that Routh's theorem is implicit in formula (8) below, as outlined in Remark 7 following the display.

Various iterations of triangles are studied in many intriguing articles, in which the transformation from a triangle in generation n to the "daughter triangle" in generation $n+1$ is seen as either a Möbius transformation or an affine mapping in a suitable "representation space". See, for instance, [2, 9, 12, 13, 15, 18, 22, 25, 31, 35, 39, 42] and references therein. The most relevant to our setting are articles [10, 21, 34], where the sequence of nested Routh's triangles T_n is considered in the case $x_n = y_n = z_n = x$ for some $x \neq 1$ and all $n \in \mathbb{Z}_+$. In that case T_n converges to the centroid of T_0. The main focus in the latter three articles is in the study of the dynamics of shapes of the triangles. In particular, in [21] necessary and sufficient conditions on x are given for the sequence T_n to be either everywhere dense or periodic in the space of shapes. Here we identify the shape of a triangle $\triangle ABC$ with a unique $\sigma \in \mathbb{C}$ such that the triangle in the complex half-plane $Im(z) \geq 0$ with vertices at

0, 1, and σ is similar to $\triangle ABC$. Thus two triangles are shape-equivalent if they are similar. We remark that the case $x_n = y_n = z_n = x$ is a particular instance of affine regularization for n-gons studied in [13, 39].

In this paper, we are mainly concerned with $T_\infty = \lim_{n\to\infty} T_n$ as defined in (4) below. In barycentric coordinates the transformation $T_n \to T_{n+1}$ is represented by a linear mapping $\mathbb{R}^3 \to \mathbb{R}^3$ associated with certain stochastic 3×3 matrix P, which is introduced in (9) below. The main technical difference between the general case and the situation when $x_n = y_n = z_n = x$ for all $n \in \mathbb{Z}_+$ is that in the latter case P is circulant and double-stochastic. The geometry of the affine map $P : \mathbb{R}^3 \to \mathbb{R}^3$ is considerably more complex, thus harder to study in the general case. However, some insight into asymptotic properties of the sequence $(T_n)_{n\in\mathbb{Z}_+}$ still can be obtained using a general theory of stochastic matrices and associated Markov chains. The present paper appears to be a first attempt in the literature to explore in this direction.

The rest of the paper is structured as follows. In Section 2, we prove basic convergence results for the triangle iterative process. In particular, we show that when $x_n = y_n = z_n$ for all $n \in \mathbb{Z}_+$, the limit T_∞ is a non-degenerate triangle if and only if $\sum_{n=0}^\infty \widetilde{x}_n < \infty$, where $\widetilde{x}_n := \min\{x_n, x_n^{-1}\}$. We also give an explicit example of nested Routh's triangles converging to a flat (collinear) triangle. In Section 3, we identify the limiting point T_∞ when $x_n = x$, $y_n = y$, and $z_n = z$ for all $n \in \mathbb{Z}_+$ and some $(x, y, z) \in (0,\infty)^3$. In Section 4, we study a sequence of nested Routh's triangles associated with a random sequence (x_n, y_n, z_n). It turns out that for any "regular" random sequence $(T_n)_{n\in\mathbb{Z}_+}$, the expected value of the random limit T_∞ coincides with the centroid of the initial triangle $\triangle A_0B_0C_0$. Finally, in Section 5, we discuss certain game-theoretic and Markov chain interpretations of a general iterative Routh's triangle sequence $(T_n)_{n\in\mathbb{Z}_+}$ and its limit point T_∞. In the process we describe a simple 3-person strategic game with the set of deterministic Nash equilibria represented by the triples $(x, y, z) \in (0,\infty)^3$ satisfying Ceva's condition $xyz = 1$.

2 Basic convergence results

We are interested in the following set:

$$T_\infty := \bigcap_{n=0}^{\infty} T_n = \lim_{m\to\infty} \bigcap_{n=0}^{m} T_n \tag{4}$$

The first identity is a formal definition of T_∞, the second one is often used to introduce the limit of a sequence of nested sets [16]. By Cantor's intersection theorem, T_∞ is a closed non-empty set.

First, we formally verify the following intuitive result:

Lemma 3. *T_∞ is either a triangle or a straight segment or a single point.*

Proof. By the Bolzano-Weierstrass theorem, the sequence $(A_n)_{n\in\mathbb{Z}_+}$ has a converging subsequence, say $(A_{n_k})_{k\in\mathbb{Z}_+}$. The sequence $(B_{n_k})_{k\in\mathbb{Z}_+}$ has a converging subsequence, say $(B_{m_k})_{k\in\mathbb{Z}_+}$. Finally, the sequence $(C_{m_k})_{k\in\mathbb{Z}_+}$ also has a converging subsequence, say $(C_{j_k})_{k\in\mathbb{Z}_+}$. Let $A := \lim_{k\to\infty} A_{j_k}$, $B := \lim_{k\to\infty} B_{j_k}$, and $C := \lim_{k\to\infty} C_{j_k}$. Then $\triangle ABC = \lim_{k\to\infty} T_{j_k}$, and hence, since the limit along a subsequence coincides with the limit of the sequence if the latter exists, $\triangle ABC = T_\infty$. Thus the limit T_∞ is a triangle, a straight segment, or a single point according to the maximal number of linearly independent points (vectors) in the set $\{A, B, C\}$. □

If the sequences x_n, y_n, and z_n are uniformly bounded away from zero and infinity, then the diameter of T_n decreases exponentially fast to zero, and therefore T_∞ is a single point. To derive this result one can use, for instance, equation (13) below. A more general sufficient condition for an inhomogeneous sequence of cevian ratios $(x_n, y_n, z_n)_{n\in\mathbb{Z}_+}$ to define nested Routh's triangles converging to a single point T_∞, together with a rate of convergence, is given in Theorem 9 below.

There are several interesting examples in the literature when an iterative sequence of triangles converges to a straight segment (*flat triangle*), see for instance [2, 18, 31]. An explicit class of iterative Routh's triangle sequences which converge to a straight segment is constructed in the following example.

Example 4. *Because of the rotation incurred at every iteration step (see Figure 1), we use here an alternate labeling of the vertices to better keep track of the limiting points. Namely, we set:*

$$\begin{array}{llll} E_n = A_n, & F_n = B_n, & G_n = C_n & \text{if} \quad n \equiv 0 \pmod 3, \\ E_n = C_n, & F_n = A_n, & G_n = B_n & \text{if} \quad n \equiv 1 \pmod 3, \\ E_n = B_n, & F_n = C_n, & G_n = A_n & \text{if} \quad n \equiv 2 \pmod 3 \end{array}$$

Heuristically, if the initial triangle $\triangle A_0B_0C_0$ is configured as in Figure 1, the E_n's are the top vertices, the F_n's the bottom right, and the G_n's the bottom left vertices.

Consider the triangle T_0 with vertices $E_0 = (0, 1)$, $F_0 = (1, 0)$, $G_0 = (0, 0)$, and two sequences of positive numbers $(t_n)_{n\in\mathbb{Z}_+}$ and $(s_n)_{n\in\mathbb{Z}_+}$ such that

$$\begin{array}{lll} 0 < t_1, & (t_n)_{n\in\mathbb{Z}_+} \textit{ is strictly increasing}, & t_n \to 1/3 \textit{ as } n \to \infty \\ 1 > s_1, & (s_n)_{n\in\mathbb{Z}_+} \textit{ is strictly decreasing}, & s_n \to 2/3 \textit{ as } n \to \infty \end{array}$$

Given $T_n = \triangle E_nF_nG_n$, we define E_{n+1}, F_{n+1} and G_{n+1} as follows:

- *Take F' to be the midpoint of G_nE_n and draw the cevian F_nF',*
- *The cevian out of G_n is the segment that intersects F_nF' at a point with abscissa s_n. This intersection point is labeled F_{n+1},*
- *The cevian out of E_n is the segment that intersects F_nG_{n+1} at a point with abscissa t_n. This intersection point is labeled G_{n+1}.*
- *The intersection of F_nF' and E_nG_{n+1} is denoted by E_{n+1}.*

Repeat the above procedure indefinitely to obtain $T_n = \triangle E_nF_nG_n$ for all $n \in \mathbb{N}$. By construction, the area of T_{n+1} is no more that half of the area of T_n. Thus T_∞ has area zero and is not a triangle. It cannot be a point because the abscissas of the G_n's are less than $1/3$ while the ones for the F_n's are greater than $2/3$. □

In the case $x_n = y_n = z_n$, the next theorem gives a necessary and sufficient condition for $(T_n)_{n\in\mathbb{Z}_+}$ to converge to a non-degenerate triangle. The definition of $\widetilde{x}_n$ in the statement of the theorem is in alignment with the fact that the shape of T_{n+1} is invariant under the "mirror" transformation of cevians $(x_n, y_n, z_n) \to \left(x_n^{-1}, y_n^{-1}, z_n^{-1}\right)$, which maps vertices of T_{n+1} into their respective isogonal conjugates in T_n.

Theorem 5. *Suppose that $z_n = y_n = x_n$ with $x_n > 0, x_n \neq 1$ for all $n \geq 0$. Let*

$$\widetilde{x}_n = \begin{cases} x_n & \text{if } \ x_n < 1, \\ x_n^{-1} & \text{if } \ x_n > 1. \end{cases}$$

If the series $\sum_{n=0}^{\infty} \widetilde{x}_n$ converges, then T_∞ is a non-degenerate triangle. If the series diverges, T_∞ is the centroid of T_0.

Proof. We note that we can take the original triangle to be an equilateral triangle without loss of generality. Indeed, if the starting triangle is scalene, we can find an invertible affine mapping $K : \mathbb{R}^2 \to \mathbb{R}^2$ of the Euclidean plane such that K transforms T_0 it into an equilateral triangle. Notice that because of the linearity of K, the centroid G of T_0 is mapped into the centroid of the equilateral triangle KT_0. It is not hard to check that affine transformations preserve cevian ratios, and hence leave the convergence mode (according to the classification given in Lemma 3) unaffected. More specifically, the above defined map K commutes with any transformation $H_n : T_0 \to T_n, n \in \mathbb{Z}_+ \cup \{\infty\}$, and in particular, $K^{-1}H_\infty KT_0 = H_\infty KT_0 = T_\infty$. This implies that the claim of the theorem is K-invariant, and we can consider KT_0 as the initial triangle.

When $x_n = y_n = z_n$ and the initial triangle T_0 is equilateral, each iterated Routh's triangle T_n, $n \in \mathbb{N}$, is also equilateral because it is symmetric with respect to rotations of 120 degrees about the centroid G of T_0. Therefore T_∞ is either a point or a triangle. To identify the limit, we look at its area Δ_∞. By virtue of (3) and because $R(x_n, x_n, x_n) = R(\widetilde{x}_n, \widetilde{x}_n, \widetilde{x}_n)$, we have

$$\frac{\Delta_\infty}{\Delta_0} = \prod_{n=0}^{\infty} \frac{(\widetilde{x}_n^3 - 1)^2}{(\widetilde{x}_n^2 + \widetilde{x}_n + 1)^3} = \prod_{n=0}^{\infty} \frac{(\widetilde{x}_n - 1)^2}{\widetilde{x}_n^2 + \widetilde{x}_n + 1} = \prod_{n=0}^{\infty} \left(1 - \frac{3\widetilde{x}_n}{\widetilde{x}_n^2 + \widetilde{x}_n + 1}\right).$$

By definition, $\widetilde{x}_n < 1$, which implies

$$1 - 3\widetilde{x}_n < 1 - \frac{3\widetilde{x}_n}{\widetilde{x}_n^2 + \widetilde{x}_n + 1} < 1 - \widetilde{x}_n.$$

Because $x \leq -\ln(1 - x)$ for $x \in (0, 1)$ we obtain

$$-\ln \frac{\Delta_\infty}{\Delta_0} > -\sum_{n=0}^{\infty} \ln\left(1 - \widetilde{x}_n\right) \geq \sum_{n=0}^{\infty} \widetilde{x}_n.$$

When $\sum_{n=0}^{\infty} \widetilde{x}_n$ diverges, the area of the limiting set T_∞ is zero and the sequence of Routh's triangles converges to a point. It remains to show that the limiting point is G, the centroid of T_0 in this case. Because of the rotational symmetry, G belongs to all the T_n's. By the uniqueness of limits, $G = T_\infty$.

In the case where $\sum_{n=0}^{\infty} \widetilde{x}_n$ converges, we choose an integer N so that $\widetilde{x}_n < 1/6$ for $n \geq N$. Using the inequality $-\ln(1-x) \leq 2x$ for $x \in (0, 1/2)$, we obtain

$$
\begin{aligned}
-\ln \frac{\Delta_\infty}{\Delta_0} &\leq -\sum_{n=0}^{N-1} \ln \frac{(\widetilde{x}_n - 1)^2}{\widetilde{x}_n^2 + \widetilde{x}_n + 1} - \sum_{n=N}^{\infty} \ln(1 - 3\widetilde{x}_n) \\
&\leq C + 6 \sum_{n=N}^{\infty} \widetilde{x}_n,
\end{aligned}
$$

where $C := -\sum_{n=0}^{N-1} \ln \frac{(\widetilde{x}_n - 1)^2}{\widetilde{x}_n^2 + \widetilde{x}_n + 1}$ is finite. We conclude that $\Delta_\infty \neq 0$, which means that the limit is a triangle. The proof of the theorem is complete. □

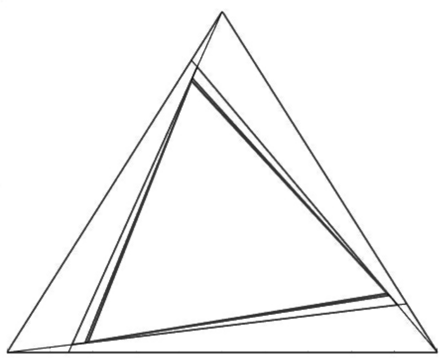

Figure 2: An illustration of iterated Routh's triangles constructed with $x_n = y_n = z_n = (n+1)!$ for n ranging from 1 to 99. The sequence $(T_n)_{n \in \mathbb{Z}_+}$ converges fast to a triangle. Although only a few triangles are easily visible, there are in fact 99 iterated triangles in this picture.

Remark 6. *The fact that the centroids of T_n and T_{n+1} coincide when $x_n = y_n = z_n$ is well known. See, for instance, Theorem 3.6 in [21]. The reduction to the equilateral triangles, which we used in the course of the proof of Theorem 5, provides a short self-contained proof of this (affine invariant) result.* □

3 Dynamical system representation

We will next describe a dynamical system representation of the nested Routh's triangles T_n. To obtain this representation, we use mass point geometry and barycentric coordinates. In these coordinates, each point O within the interior of T_0 is described using a (unique) triple of non-negative numbers (α, β, γ) such that, considering points on the plane as vectors, $O = \alpha A_0 + \beta B_0 + \gamma C_0$ and $\alpha + \beta + \gamma = 1$. Notice that O is the center of mass of $\triangle A_0 B_0 C_0$ if for some $k > 0$ the mass $k\alpha$ is put at the vertex A, the mass $k\beta$ at B, and the mass $k\gamma$ at C_0.

If we put weights $(y_n, y_n z_n, 1)$ at the vertexes (A_n, B_n, C_n) of the triangle T_n, then the intersection A_{n+1} of the lines $B_n B_n'$ and $C_n C_n'$ will be the center of mass of the triangle T_n. Identifying A_k, B_k and C_k with vectors starting at the origin and ending at the points denoted by the corresponding capital letters, we therefore obtain

$$
A_{n+1} = \frac{1}{1 + y_n + y_n z_n} \Big(y_n \cdot A_n + y_n z_n \cdot B_n + C_n \Big). \tag{5}
$$

Similarly,

$$B_{n+1} = \frac{1}{1+z_n+x_nz_n}\Big(A_n + z_n\cdot B_n + x_nz_n\cdot C_n\Big), \tag{6}$$

$$C_{n+1} = \frac{1}{1+x_n+x_ny_n}\Big(x_ny_n\cdot A_n + B_n + x_n\cdot C_n\Big). \tag{7}$$

For formal 3-vectors, whose components are points in the plane within T_0, this can be put in a formal vector-matrix equation form as follows:

$$\big(A_{n+1}, B_{n+1}, C_{n+1}\big)^T = M_n\big(A_n, B_n, C_n\big)^T, \tag{8}$$

where the superscript ^{T} indicates that the triple is transposed, i. e. converted from a row to a column, and

$$M_n := \begin{pmatrix} \frac{y_n}{1+y_n+y_nz_n} & \frac{y_nz_n}{1+y_n+y_nz_n} & \frac{1}{1+y_n+y_nz_n} \\ \frac{1}{1+z_n+x_nz_n} & \frac{z_n}{1+z_n+x_nz_n} & \frac{x_nz_n}{1+z_n+x_nz_n} \\ \frac{x_ny_n}{1+x_n+x_ny_n} & \frac{1}{1+x_n+x_ny_n} & \frac{x_n}{1+x_n+x_ny_n} \end{pmatrix}. \tag{9}$$

Remark 7. *Notice that, in accordance with Routh's theorem (see, for instance, [11] or [7]),* $\det M_n = R(x_n, y_n, z_n)$. *In fact, it follows from* (5) *and the analogous formulas for* B_{n+1} *and* C_{n+1} *that (cf. [10, 20, 27, 36])*

$$M_n = \frac{1}{\Delta_n}\begin{pmatrix} \mathrm{Area}(\triangle A_{n+1}B_nC_n) & \mathrm{Area}(\triangle A_{n+1}C_nA_n) & \mathrm{Area}(\triangle A_{n+1}A_nB_n) \\ \mathrm{Area}(\triangle B_{n+1}B_nC_n) & \mathrm{Area}(\triangle B_{n+1}C_nA_n) & \mathrm{Area}(\triangle B_{n+1}A_nB_n) \\ \mathrm{Area}(\triangle C_{n+1}B_nC_n) & \mathrm{Area}(\triangle C_{n+1}C_nA_n) & \mathrm{Area}(\triangle C_{n+1}A_nB_n) \end{pmatrix}.$$

In particular, $\mathrm{tr}\, M_n = 1 - R(x_n, y_n, z_n)$. *We remark that the characteristic equation associated with* M_n *is* $\lambda^3 - \lambda^2\big(1 - R(x_n, y_n, z_n)\big) - R(x_n, y_n, z_n) = 0$. *Hence, the eigenvalues of* M_n *are the Perron-Frobenius value one and, in addition, two complex roots of the quadratic equation* $\lambda^2 + \lambda R(x_n, y_n, z_n) + R(x_n, y_n, z_n) = 0$. *These observations indicate that when cevian ratios* (x_n, y_n, z_n) *are chosen at random and form a stationary ergodic sequence, the distribution of the real-valued random variable* $R(x_n, y_n, z_n)$ *can serve to measure a "random capacity" of the iterative triangle process. The distribution is in principle directly available from the input data, the joint distribution of the cevian ratios* (x_n, y_n, z_n). □

It follows by induction that for a general $n \in \mathbb{N}$,

$$\big(A_n, B_n, C_n\big)^T = M_{n-1}M_{n-2}\cdots M_0\big(A_0, B_0, C_0\big)^T \tag{10}$$

We remark that equations (5)-(8) and (10) can be alternatively interpreted as identities for the complex numbers A_n, B_n, C_n instead of the corresponding real vectors.

For $n \in \mathbb{Z}_+$, let

$$a_n = \overrightarrow{B_nC_n}, \qquad b_n = \overrightarrow{C_nA_n}, \qquad c_n = \overrightarrow{A_nB_n}, \tag{11}$$

and

$$\begin{aligned} u_n &= \frac{1 - x_n y_n z_n}{(1 + z_n + x_n z_n)(1 + x_n + x_n y_n)}, \\ v_n &= \frac{1 - x_n y_n z_n}{(1 + y_n + y_n z_n)(1 + x_n + x_n y_n)}, \\ w_n &= \frac{1 - x_n y_n z_n}{(1 + y_n + y_n z_n)(1 + z_n + x_n z_n)}. \end{aligned} \tag{12}$$

It follows from (5)-(7) that

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -x_n u_n & u_n \\ v_n & 0 & -y_n v_n \\ -z_n w_n & w_n & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}. \tag{13}$$

The matrix in (13) is the cofactor matrix of M_n, and hence

$$(a_{n+1}, b_{n+1}, c_{n+1}) = R(x_n, y_n, z_n)^{-1} \cdot (a_n, b_n, c_n) M_n^{-1}.$$

Using the law of cosines, we deduce from (13) that

$$\big(|a_{n+1}|^2, |b_{n+1}|^2, |c_{n+1}|^2\big)^T = Q_n \big(|a_n|^2, |b_n|^2, |c_n|^2\big)^T, \tag{14}$$

where

$$Q_n := \frac{1}{2} \begin{pmatrix} x_n u_n^2 & (2x_n^2 - x_n) u_n^2 & (2 - x_n) u_n^2 \\ (2 - y_n) v_n^2 & y_n v_n^2 & (2y_n^2 - y_n) v_n^2 \\ (2z_n^2 - z_n) w_n^2 & (2 - z_n) w_n^2 & z_n w_n^2 \end{pmatrix}.$$

We will exploit (14) in Section 4 below.

The following is the main result of this section.

Theorem 8. *Suppose that $x_n = x, y_n = y, z_n = z$ for some $x, y, z > 0$ such that $xyz \neq 1$ and all $n \in \mathbb{Z}_+$. Then*

$$T_\infty = \frac{\theta_1}{\theta_1 + \theta_2 + \theta_3} A_0 + \frac{\theta_2}{\theta_1 + \theta_2 + \theta_3} B_0 + \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} C_0, \tag{15}$$

where

$$\begin{aligned} \theta_1 &= (xy(1 + xz) + 1)(1 + y + yz), \\ \theta_2 &= (yz(1 + yx) + 1)(1 + z + xz), \\ \theta_3 &= (zx(1 + zy) + 1)(1 + x + xy). \end{aligned} \tag{16}$$

Proof. Let M denote the common value of the matrices M_n introduced in (9). Then (10) becomes $\big(A_n, B_n, C_n\big)^T = M^n \big(A_0, B_0, C_0\big)^T$. It is easy to verify that the vector $\overrightarrow{\theta} := (\theta_1, \theta_2, \theta_3)^T$ is a left eigenvector of the matrix M, namely

$$\overrightarrow{\theta} = \overrightarrow{\theta} M. \tag{17}$$

For $i = 1, 2, 3$, let

$$\pi_i = \theta_i(\theta_1 + \theta_2 + \theta_3)^{-1}. \tag{18}$$

By virtue of (17), the probability vector $\overrightarrow{\pi} := (\pi_1, \pi_2, \pi_3)$ represents the stationary distribution of a Markov chain which evolves according to the transition kernel M. It follows from (17) that M^n converges as $n \to \infty$ to a matrix whose columns coincide and all three are equal to $(\pi_1, \pi_2, \pi_3)^T$. In the language of the Markov chains theory, the latter statement is the claim that stationary distribution is also the limiting distribution of the Markov chain. The claim is true because M is a strictly positive matrix, and hence the associated Markov chain is ergodic (see, for instance, [6, 37] for details). Thus

$$\begin{aligned} \lim_{n\to\infty} M^n\big(A_0, B_0, C_0\big)^T &= \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \\ &= \begin{pmatrix} \pi_1 A_0 + \pi_2 B_0 + \pi_3 C_0 \\ \pi_1 A_0 + \pi_2 B_0 + \pi_3 C_0 \\ \pi_1 A_0 + \pi_2 B_0 + \pi_3 C_0 \end{pmatrix}. \end{aligned} \tag{19}$$

The proof is complete. □

Our next result gives a necessary condition for an inhomogeneous sequence of cevian ratios $(x_n, y_n, z_n)_{n\in\mathbb{Z}}$ to define nested Routh's triangles converging to a single point T_∞ together with a rate of convergence.

Theorem 9. *Let $\xi_n := \max_i \min_j M_n(i,j)$. If $\sum_{n=0}^{\infty} \xi_n = \infty$, then T_∞ is a single point and for any $n \in \mathbb{N}$,*

$$\begin{aligned} &\max\{|\overrightarrow{A_nT_\infty}|, |\overrightarrow{B_nT_\infty}|, |\overrightarrow{C_nT_\infty}|\} \\ &\qquad \le \prod_{k=0}^{n-1}(1-\xi_k) \cdot \max\{|A_0|, |B_0|, |C_0|\}. \end{aligned} \tag{20}$$

The result stated in the theorem is merely a rephrasing in our context of a well-known result for a product of stochastic matrices which has been stated in various forms in many papers and monographs. See, for instance, Section 2.A.2 in [37], [41], [1], or [6, Lemma 9] where different, suitable for general non-negative matrices, bounds ξ_i are used. In a similar form, with the same ξ_i, the above result is explicitly stated in [40, Lemma 3.3]. For the reader's convenience, we will next outline a short proof of Theorem 9.

Let $i^* \in \{1, 2, 3\}$ be a state such that $\xi_n = \min_j M_n(i^*, j)$. By the assumptions of Theorem 9, matrix M_n satisfies Deoblin's condition, namely $M_n(i,j) \ge \xi_n \delta^*(j)$, where δ^* is a probability vector in dimension 3 such that $\delta^*(j) = 1$ if $j = i^*$ and $\delta^*(j) = 0$ otherwise. Thus the convergence of the backward products $M_{n-1} \cdots M_0$ to a column stochastic matrix M_∞ follows, for example, from [30, Theorem A]. Each row of M_∞ form the same probability vector, say $\pi = (\pi_1, \pi_2, \pi_3)$. A contraction property of Doeblin's stochastic kernels (in particular, strictly positive stochastic matrices), see [14, p. 197], implies then that

$\|\nu M_n - \pi M_n\| \le (1-\xi_n)\|\nu - \pi\|$, where ν is any probability vector in dimension 3 and $\|\cdot\|$ is the total variation norm. Thus (cf. [37, 41])

$$\max_j \sum_{i=1}^{3} \big| M_{n-1}\cdots M_0(i,j) - M_\infty(i,j)\big| \le \prod_{k=0}^{n-1}(1-\xi_k),$$

which implies (20) with $T_\infty = \pi_1 A + \pi_2 B + \pi_3 C$ by triangle inequality.

4 Random iterations

In this section we consider the iteration of Routh's triangles associated with random cevian ratios $(x_n, y_n, z_n)_{n\in\mathbb{Z}_+}$.

Proposition 10. *Suppose cevian ratios $(x_n, y_n, z_n)_{n\in\mathbb{Z}_+}$ are sampled independently and identically from a joint distribution defined on $(0,\infty)^3$. Then T_∞ is a single point with probability one.*

Proof. The conclusion follows readily if $P(x_n y_n z_n = 1) > 0$. Thus we will assume that $x_n y_n z_n \ne 1$ with probability one. Let $d_n = \operatorname{diam}(T_n)$, and recall that the diameter of a triangle is the length of its longest side. Thus we need to show that $d_n \to 0$, as $n \to \infty$, under the conditions of the theorem. Let $\gamma_n = \dfrac{d_n}{d_{n-1}}$. By the law of large numbers, with probability one,

$$\lim_{n\to\infty} \frac{1}{n} \ln\Big(\prod_{k=1}^{n} \gamma_n\Big) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \gamma_n = E(\ln \gamma_1),$$

where E denotes the expected value. Since the inclusion $T_n \subset T_{n-1}$ is strict, we know that $P(\ln \gamma_1 < 0) = 1$. Therefore, $E(\ln \gamma_1) = -a$ for some $a > 0$. With probability one, for all n sufficiently large (how large depends on the random realization of the sequence d_n), we have

$$\frac{1}{n} \ln \frac{d_n}{d_0} = \frac{1}{n} \ln\Big(\prod_{k=1}^{n} \gamma_n\Big) < -\frac{a}{2}, \quad \text{and hence} \quad d_n < d_0 \exp\Big(-\frac{na}{2}\Big).$$

Thus we have shown that $d_n \to 0$ as $n \to \infty$. The proof is complete. □

Remark 11. *The assumption of independence in the conditions of Proposition 10 can be replaced by a weaker condition that the cevian triples form a stationary and ergodic sequence. The above proof goes through verbatim with Birkhoff's ergodic theorem [16] taking the place of the classical law of large numbers. Note that, in view of the Poincaré's recurrence theorem [16], if $P(x_n y_n z_n = 1 > 0)$ then the number of iterations before reaching a degenerate triangle (a point) is finite with probability one. We remark that an alternative proof of Proposition 10 (or its extension to a stationary and ergodic sequence of cevian triples) can be given by applying the result in Theorem 9 to a stationary and ergodic sequence ξ_n which is introduced in the statement of the theorem.* □

Recall that the distribution of a random vector $(x_n, y_n, z_n) \in \mathbb{R}^3$ is called exchangeable if for any permutation of (x_n, y_n, z_n) the joint probability distribution of the permuted triple is the same as the joint probability distribution of the original one [16]. The following theorem is the main result of this section. It can be considered as a probabilistic counterpart of Theorem 5.

Theorem 12. *Under the conditions of Proposition 10, if the common distribution of the vectors (x_n, y_n, z_n), $n \in \mathbb{Z}_+$, is exchangeable then the expected value $E(T_\infty)$ is the centroid of T_0.*

Proof. Taking the expectation in (10) and using the linearity property of the expectation one can show by induction that for any $n \in \mathbb{N}$,

$$\big(E(A_n), E(B_n), E(C_n)\big)^T = K^n \big(A_0, B_0, C_0\big)^T,$$

where $K := E(M_n)$ is a matrix whose entries are the expected values of the corresponding entries of M_n. Since the distribution of (x_n, y_n, z_n) is exchangeable, K is a double-stochastic matrix, that is the sum of the entries in each row and column is one. The claim follows from the fact that the probability vector $\frac{1}{3}(1,1,1)^T$ is the left Perron-Frobenius eigenvector of K, and hence (compare to (19) and see the discussion above it)

$$\lim_{n\to\infty} K^n = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

The proof of the theorem is complete. □

Recall a_n, b_n, c_n from (11). We conclude this section with the following theorem collecting several results on the rate of convergence of a random sequence T_n to a random limit point T_∞. For $n \in \mathbb{Z}_+$, let

$$\chi_n = E(\Delta_n) \qquad \text{and} \qquad \eta_n = \max\{|a_n|^2, |b_n|^2, |c_n|^2\}$$

Theorem 13.

(i) Assume that $(x_n, y_n, z_n)_{n\in\mathbb{Z}_+}$ is a stationary and ergodic random sequence. Then, with probability one,

$$\lim_{n\to\infty} \frac{1}{n} \log \Delta_n = E\big(\log R(x_0, y_0, z_0)\big), \tag{21}$$

$$\lim_{n\to\infty} \frac{1}{n} \log \eta_n = \lambda, \tag{22}$$

for some $\lambda < 0$.

(ii) If $(x_n, y_n, z_n)_{n\in\mathbb{Z}_+}$ is an i. i. d. sequence, then

$$\chi_n = \big\{E\big(R(x_0, y_0, z_0)\big)\big\}^n \Delta_0, \qquad n \in \mathbb{N}, \tag{23}$$

$$\lim_{n\to\infty} \frac{1}{n} \log E(|a_n|^2) = \lim_{n\to\infty} \frac{1}{n} \log E(|b_n|^2) = \lim_{n\to\infty} \frac{1}{n} \log E(|c_n|^2) = \delta \tag{24}$$

for some $\delta < 0$.

(iii) Under the conditions of Proposition 10, δ in (24) is given by

$$\delta = \log E\big(u_0^2(x_0^2 - x_0 + 1)\big), \tag{25}$$

where u_0 is introduced in (12). In fact, in this case,

$$\begin{aligned} \lim_{n\to\infty} e^{-n\delta} E(|a_n|^2) &= \lim_{n\to\infty} e^{-n\delta} E(|b_n|^2) = \lim_{n\to\infty} e^{-n\delta} E(|c_n|^2) \\ &= \frac{1}{3} E\big(|a_n|^2 + |b_n|^2 + |c_n|^2\big) \end{aligned} \tag{26}$$

Furthermore, if in addition there exists $\varepsilon > 0$ such that $P\big(M_0(i,j) \in (\varepsilon, 1-\varepsilon)\big) = 1$ for all $i, j \in \{1,2,3\}$ (for instance, with probability one, x_0, y_0, and z_0 are uniformly bounded away from zero and from infinity), then (22) can be strengthened to

$$\lim_{n\to\infty} \frac{1}{n} \log |a_n| = \lim_{n\to\infty} \frac{1}{n} \log |b_n| = \lim_{n\to\infty} \frac{1}{n} \log |c_n| = \frac{\lambda}{2} \tag{27}$$

Proof.

(i) The result in (21) follows immediately from the Birkhoff ergodic theorem and Routh's theorem. The result in (22) for some $\lambda < 0$ is a direct implication of (14) along with the Furstenberg-Kesten theorem for products of random matrices [17]. The same argument as we used to establish Proposition 10 (see also Remark 11) shows that $\lambda < 0$.

(ii) The identity in (23) is evident from Routh's theorem. The limit result in (24) for some $\delta < 0$ follows from (14) by taking the expectation on both the sides of the equation. Moreover, $\delta = \log \lambda_Q$ where $\lambda_Q > 0$ is the Perron-Frobenius eigenvalue of the 3×3 matrix $H := E(Q_0)$ whose entries are expectations of the corresponding entries of Q_0. Indeed, for some positive reals f_1, f_2, f_3 and $f := (f_1, f_2, f_3) \in \mathbb{R}^3$ we have $Hf = \lambda_Q f$. Then by virtue of (14) for any real constant $c > 0$ such that

$$c^{-1} f_1 < E(|a_n|^2) < c f_1,\ c^{-1} f_2 < E(|b_n|^2) < c f_2,\ c^{-1} f_3 < E(|c_n|^2) < c f_3,$$

we have (the vector inequalities below is a notation to denote that the corresponding inequalities hold component-wise)

$$\begin{aligned} c^{-1}\lambda_Q^n f < \big(E(|a_n|^2), E(|a_n|^2), E(|a_n|^2)\big)^T = H^n \big(|a_0|^2, |b_0|^2, |c_0|^2\big)^T < \\ < c\lambda_Q^n f, \end{aligned}$$

from which the claim in (24) readily follows with $\delta = \log \lambda_Q$.

(iii) Recall that $\delta = \log \lambda_Q$. It is not hard to verify that under the conditions of Proposition 10, we have $\lambda_Q = E\big(u_n^2(x_n^2 - x_n + 1)\big)$, which is the sum of the elements in a row of $H := E(Q_n)$. It follows that $\lambda_Q^{-1} H$ is a double-stochastic matrix, and hence

$$\begin{aligned} &\lim_{n\to\infty} e^{-n\delta}\big(E(|a_n|^2), E(|b_n|^2), E(|c_n|^2)\big)^T \\ &= \lim_{n\to\infty} (\lambda_Q^{-1} H)^n \big(E(|a_0|^2), E(|b_0|^2), E(|c_0|^2)\big)^T \\ &= \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \big(E(|a_0|^2), E(|b_0|^2), E(|c_0|^2)\big)^T. \end{aligned}$$

Finally, (27) follows from the Corollary on p. 462 of [17]. □

Heuristically, Theorem 13 states that the rate of convergence of the sequence T_n to the limit T_0 is exponential. We conclude this section with the observation that in addition to the results listed in the theorem, (20) implies that when $(x_n, y_n, z_n)_{n\in\mathbb{Z}_+}$ and hence also $(\xi_n)_{n\in\mathbb{Z}_+}$ are stationary and ergodic sequences,

$$\limsup_{n\to\infty} \frac{1}{n} \log\big[\max\{|\overrightarrow{A_nT_\infty}|, |\overrightarrow{B_nT_\infty}|, |\overrightarrow{C_nT_\infty}|\}\big] \le E(\log \xi_0) < 0$$

5 A job allocation interpretation

The goal of this section is to describe two job allocation procedures connected to the Routh's triangle iteration process. The second one provides a curious interpretation of the limit T_∞ in Theorem 8.

First, we observe that cevian ratios (x, y, z) satisfying the condition $xyz = 1$ can be identified with deterministic Nash equilibria in the following 3-person game. The underlying idea is expressed by the identities in (28) below, and it utilizes a geometric lemma which is referred to in [20] as an "area principle".

Suppose that players Alice (A), Bob (B), and Colette (C) have a job assignment to complete, and they want to divide the workload fairly. Assume that the total amount of work to be done is one unit and denote the portion of the work allocated to a player X by W_X, where $X \in \{A, B, C\}$. The respective strategies of the players A, B, C are positive reals x, y, z. The strategies are interpreted as a suggestion of the player with regard to how the portion of the work which is allocated to the two other players should be divided. More specifically, if the triple (x, y, z) satisfies Ceva's condition $xyz = 1$ and the associated cevians intersect at a point O within T_0, the work is divided between the players in such a way that

$$W_A = \text{Area}(\triangle B_0C_0O),\ \ W_B = \text{Area}(\triangle C_0A_0O),\ \ W_C = \text{Area}(\triangle A_0B_0O),$$

and hence [20]

$$\frac{W_B}{W_C} = x, \qquad \frac{W_C}{W_A} = y, \qquad \frac{W_A}{W_B} = z. \tag{28}$$

If Ceva's condition is not satisfied, we look at the iterative triangle process associated with the cevian ratios $(x_n, y_n, z_n) = (x, y, z)$ for all $n \in \mathbb{Z}_+$ and set

$$\begin{aligned} W_A &= \sum_{n=0}^{\infty} \text{Area}(\triangle B_nC_nA_{n+1}) = \frac{y}{1+y+yz} \sum_{n=0}^{\infty} R(x,y,z)^n \\ &= \frac{y}{1+y+yz} \cdot \frac{1}{1-R(x,y,z)}, \end{aligned}$$

and, similarly,

$$\begin{aligned} W_B &= \sum_{n=0}^{\infty} \text{Area}(\triangle C_nA_nB_{n+1}) = \frac{z}{1+z+xz} \cdot \frac{1}{1-R(x,y,z)}, \\ W_C &= \sum_{n=0}^{\infty} \text{Area}(\triangle A_nB_nC_{n+1}) = \frac{x}{1+x+xy} \cdot \frac{1}{1-R(x,y,z)}. \end{aligned}$$

Assume that each player $X \in \{A, B, C\}$ minimizes their workload W_X. Recall that a vector of strategies (x, y, z) is called a Nash equilibrium of the above game if no player benefits from a unilateral change when the remaining two players maintain their strategies fixed [19]. Clearly, a triple $(x, y, z) \in (0, \infty)^3$ is a Nash equilibrium if and only if $R(x, y, z) = 0$, that is $xyz = 1$.

Consider now the following modification of the above game-theoretic framework. Assume that a certain job, say cleaning a rented apartment, must be done on a daily basis and takes one person to accomplish. Three roommates, Alice, Bob, and Colette want to devise a long-term schedule. For $n \in \mathbb{Z}_+$, let X_n be the person assigned to the job at day n. For $X \in \{A, B, C\}$ let

$$\pi_X = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \mathbf{1}(X_k = X), \tag{29}$$

provided that the above limit exists. As before, the strategies x, y, z serve as "recommendations" from the players on how the job should be divided. Specifically, the roommates assign the cleaning job at random so that X_n is a Markov chain with a 3×3 transition kernel $P(X_{n+1} = j|X_n = i) = M(i, j)$, where M is M_n defined in (9) with $x_n = x, y_n = y, z_n = z$, and we identify A with state 1, B with state 2, and C with state 3 of the Markov chain. By the ergodic theorem, the limits in (29) exist and, moreover,

$$\pi_A(x, y, z) = \pi_1, \qquad \pi_B(x, y, z) = \pi_2, \qquad \pi_C(x, y, z) = \pi_3, \tag{30}$$

where π_i, $i = 1, 2, 3$, are defined in (18). In particular, if the triple (x, y, z) satisfies Ceva's condition, we have $\pi_B/\pi_C = x$, $\pi_C/\pi_A = y$, and $\pi_A/\pi_B = z$. It turns out that if the goal of each player is to minimize the asymptotic average workload, the only deterministic Nash equilibrium in the game is $(x, y, z) = (1, 1, 1)$.

Theorem 14. *Consider a 3-person game where the strategy of a player $X \in \{A, B, C\}$ is a positive real number and the (negative) payoff associated with a vector of strategies $(x, y, z) \in (0, \infty)^3$ is $-\pi_X(x, y, z)$, with π_X given by* (30). *Then the only deterministic Nash equilibrium in the game is the "fair division" $x = y = z = 1$.*

Proof. Suppose that (x, y, z) is a Nash equilibrium. Assume in addition that $xyz \neq 1$. Since player A prefers x over $y^{-1}z^{-1}$, we have $\pi_A \le \frac{y}{1+y+yz}$. Similarly, it must be the case that $\pi_B \le \frac{z}{1+z+xz}$ and $\pi_C \le \frac{x}{1+y+xy}$. But then

$$\begin{aligned} 1 &= \pi_A + \pi_B + \pi_C \le \frac{y}{1+y+yz} + \frac{z}{1+z+xz} + \frac{x}{1+y+xy} \\ &= 1 - R(x, y, z), \end{aligned}$$

which is only possible if $R(x, y, z) = 0$. The latter assertion however contradicts the assumption $xyz \neq 1$.

Thus we can assume without loss of generality that $xyz = 1$. Since player A prefers $x = y^{-1}z^{-1}$ over any other value of $x > 0$, it must be the case that for all $u > 0$,

$$\begin{aligned} &\frac{y}{1+y+yz} \le \\ &\le \frac{(uy(1+uz)+1)(1+y+yz)}{uy(1+uz)+1)(1+y+yz)+(yz(1+yu)+1)(1+z+uz)+(zu(1+zy)+1)(1+u+uy)} \end{aligned} \tag{31}$$

Since $u = y^{-1}z^{-1}$ is a minimizer of the right-hand side, the derivative of the expression in the right hand side with respect to u is equal to zero at $u = y^{-1}z^{-1}$. A little algebra shows that the latter condition is equivalent to $x = y^{-1}z^{-1} = 1$. Using similar arguments for π_B and π_C, one can deduce that $y = z = 1$.

So far we have shown that if there is a Nash equilibrium, it must be $(1, 1, 1)$. To complete the proof it remains to verify that $(x, y, z) = (1, 1, 1)$ is indeed a Nash equilibrium. Instead of checking second order derivatives, we will directly verify (31) with $(y, z) = (1, 1)$. To this end it suffices to show that

$$\frac{1}{3} \leq \frac{3(1 + u + u^2)}{3(1 + u + u^2) + (2 + u)^2 + (2u + 1)^2}$$

It is easy to see that the last inequality is equivalent to the trivial $(u-1)^2 \geq 0$, and therefore holds true. The proof of the theorem is complete. □

Acknowledgements

We would like to thank the anonymous referee for their useful comments, in particular for pointing out references [13, 39].

References

[1] J. M. Anthonisse, H. Tijms, Exponential convergence of products of stochastic matrices, *J. Math. Anal. Appl.* **59** (1977), 360–364.

[2] I. Bárány, A. F. Beardon, T. K. Carne, Barycentric subdivision of triangles and semigroups of Möbius maps, *Mathematika* **43** (1996), 165–171.

[3] J. Benitez, A unified proof of Ceva and Menelaus' theorems using projective geometry, *J. Geom. Graph.* **11** (2007), 39–44.

[4] Á. Bényi and Ć. Branko, Triangles and groups via cevians, *J. Geom.* **103** (2012), 375–408.

[5] Á. Bényi and Ć. Branko, A generalization of Routh's triangle theorem, *Amer. Math. Monthly* **120** (2013), 841–846.

[6] E. Bolthausen and I. Goldsheid, Recurrence and transience of random walks in random environments on a strip, *Comm. Math. Phys.* **214** (2000), 429–447.

[7] O. Bottema, The area of a triangle as a function of the barycentric coordinates of its vertices. In *Topics in Elementary Geometry*, 2nd edition, Springer, 2008, pp. 29–32.

[8] P. Bougerol, J. Lacroix, *Products of Random Matrices with Applications to Schrödinger Operators*, Progress in Probability and Statistics, vol. 8, Birkhäuser, 1985.

[9] S. Butler, R. Graham, Subdivision using angle bisectors is dense in the space of triangles, *Amer. Math. Monthly* **120** (2013), 622–630.

[10] G. Z. Chang, Ph. J. Davis, Iterative processes in elementary geometry, *Amer. Math. Monthly* **90** (1983), 421–431.

[11] H. S. M. Coxeter, *Introduction to Geometry*, second edition, John Wiley & Sons, Inc., 1989.

[12] M. De Villiers, Over and over again: two geometric iterations with triangles, *Learning and Teaching Mathematics* **16** (2014), 40–45.

[13] S. Donisi, H. Martini, G. Vincenzi, and G. Vitale, *Polygons derived from polygons via iterated constructions*, Differ. Geom. Dyn. Syst. **18** (2016), 14–31.

[14] J. L. Doob, *Stochastic Processes*, Wiley-Interscience, Revised edition, New York, 1990.

[15] D. Downes, 99.14 A recurrence relation for the limit point of nested triangles, *Mth. Gaz.* **99**, no. 545 (2015), 323–325.

[16] R. Durret, *Probability: Theory and Examples*, fourth edition, Cambridge University Press, 2000.

[17] H. Furstenberg and H. Kesten, Products of random matrices, *Ann. Math. Statist.* **31** (1960), 269–556.

[18] J. Gates, The shape of a sequence of dual random triangles, *Bernoulli* **12** (2006), 55–63.

[19] R. Gibbons, *Game Theory for Applied Economists*, Princeton University Press, 1992.

[20] B. Grünbaum, G. C. Shephard, Ceva, Menelaus and the area principle, *Math. Mag.* **68** (1995), 254–268.

[21] M. Hajja, On nested sequences of triangles, *Results Math.* **54** (2009), 289–299.

[22] M. Hajja, Generalized Napoleon and Torricelli transformations and their iterations, *Beitr. Algebra Geom.* **51** (2010), 171–190.

[23] W. J. Heiser, Geometric representation of association between categories, *Psychometrika* **69** (2004), 513–545.

[24] J. B. Hogendijk, Al-Mutaman ibn Hüd, 11the century king of Saragossa and brilliant mathematician, *Historia Math.* **22** (1995), 1–18.

[25] D. Ismailescu, J. Jacobs, On sequences of nested triangles, *Period. Math. Hungar.* **53** (2006), 169–184.

[26] M. S. Klamkin, A. Liu, Three more proofs of Routh's theorem, *Crux Mathematicorum* **7** (1981), 199–203.

[27] J. S. Kline, D. Velleman, Yet another proof of Routh's theorem, *Crux Mathematicorum* **21** (1995), 37–40.

[28] J. Kozma, Á. Kurusa, Cevas and Menelaus theorems characterize the hyperbolic geometry among Hilbert geometries, *J. Geom.* **106** (2015), 465–470.

[29] S. Landy, A Generalization of Ceva's theorem to higher dimensions, *Amer. Math. Monthly* **95** (1988), 936–939.

[30] A. Leizarowitz, On infinite products of stochastic matrices, Linear Algebra Appl. **168** (1992), 189–219.

[31] D. Mannion, Convergence to collinearity of a sequence of random triangle shapes, *Adv. in Appl. Probab.* **22** (1990), 831–844.

[32] F. Marko and S. Litvinov, Routh's theorem for simplices, preprint, 2014, available electronically at `https://arxiv.org/abs/1408.1929`.

[33] L. A. Masal'tsev, Incidence theorems in spaces of constant curvature, *Int. J. Math. Math. Sci.* **72** (1994), 3201–3206.

[34] H. Nakamura, K. Oguiso, Elementary moduli space of triangles and iterative processes, *J. Math. Sci. Univ. Tokyo* **10** (2003), 209–224.

[35] G. Nicollier, Dynamics of the nested triangles formed by the tops of the perpendicular bisectors, *Forum Geometricorum*, **14** (2014), 31–41.

[36] I. Niven, A new proof of Routh's theorem, *Math. Mag.* **49** (1976), 25–27.

[37] A. Paz, *Introduction to Probabilistic Automata* (Computer Science and Applied Mathematics Series), Academic Press, 1971.

[38] J. Richter-Gebert, Meditations on Ceva's theorem, In *The Coxeter Legacy: Reflections and Projections*, C. Davis and E. W. Ellers Editors, AMS and Fields Institute, 2005, pp. 227–254.

[39] O. Röschel, *Polygons and iteratively regularizing affine transformations*, Beitr. Algebra Geom. **58** (2017), 69–79.

[40] A. Roitershtein, A note on multitype branching processes with immigration in a random environment, *Ann. Probab.* **35** (2007), 1573–1592.

[41] E. Seneta, *Non-negative Matrices and Markov Chains* (Springer Series in Statistics), 2nd Edition, Springer, 2013.

[42] A. Wares, Conjectures pertaining to repeated partitioning of a triangle, *Internat. J. Math. Ed. Sci. Tech.* **39** (2008), 814–819.

[43] P. Wernicke, The theorems of Ceva and Menelaus and their extension, *Amer. Math. Monthly* **34** (1927), 468–472.