

Random walks in a sparse random environment

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Abstract

We introduce random walks in a sparse random environment on \mathbb{Z} and investigate basic asymptotic properties of this model, such as recurrence-transience, asymptotic speed, and limit theorems in both the transient and recurrent regimes. The new model combines features of several existing models of random motion in random media and admits a transparent physical interpretation. More specifically, a random walk in a sparse random environment can be characterized as a “locally strong” perturbation of a simple random walk by a random potential induced by “rare impurities,” which are randomly distributed over the integer lattice. Interestingly, in the critical (recurrent) regime, our model generalizes Sinai’s scaling of $(\log n)^2$ for the location of the random walk after n steps to $(\log n)^\alpha$, where $\alpha > 0$ is a parameter determined by the distribution of the distance between two successive impurities. Similar scaling factors have appeared in the literature in different contexts and have been discussed in [28] and [30].

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1 Introduction: The model

We start with a general description of one-dimensional random walks in a random environment. Let $\Omega = (0, 1)^\mathbb{Z}$ and let \mathcal{F} be the Borel σ -algebra of subsets of the product space Ω . A *random environment* is a random element $\omega = (\omega_n)_{n \in \mathbb{Z}}$ of the measurable space (Ω, \mathcal{F}) . The environment determines the transition kernel of the underlying random walk. Namely, a *random walk in a random environment* $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Omega$ is a Markov chain $(X_n)_{n \geq 0}$ on \mathbb{Z} , the transition kernel of which is given by

$$P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} \omega_i & \text{if } j = i + 1 \\ 1 - \omega_i & \text{if } j = i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

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The probability distribution of this Markov chain, starting at the initial state $X_0 = x$, $x \in \mathbb{Z}$, is denoted by $P_{x,\omega}$ and is referred to as the *quenched law* of the random walk.

We denote the probability distribution of ω in (Ω, \mathcal{F}) by P , and we let E_P denote the corresponding expectation operator. That is, for a measurable function $f(\omega)$ of the environment ω , we have $E_P(f) = \int_{\Omega} f(\omega) dP(\omega)$. Let \mathcal{G} be the cylinder σ -algebra on $\mathbb{Z}^{\mathbb{N}}$. A *random walk in a random environment* (RWRE) associated with P is a process (X, ω) on the measurable space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathcal{F} \otimes \mathcal{G})$ equipped with the *annealed* probability law $\mathbb{P} = P \otimes P_{\omega}$, which is defined by

$$\mathbb{P}_x(F \times G) = \int_{\mathcal{F}} P_{x,\omega}(G) P(d\omega), \quad F \in \mathcal{F}, G \in \mathcal{G}.$$

The expectations under the laws $P_{x,\omega}$ and \mathbb{P}_x are denoted by $E_{x,\omega}$ and \mathbb{E}_x , respectively. We will usually omit the index 0 when $x = 0$, which is to say that we will write P_{ω} , E_{ω} , \mathbb{P} , and \mathbb{E} for $P_{0,\omega}$, $E_{0,\omega}$, \mathbb{P}_0 , and \mathbb{E}_0 , respectively. Notice that, since the process “learns” the environment according to the Bayes rule as time progresses, $X = (X_n)_{n \geq 0}$ is not, in general, a Markov chain under the annealed measure \mathbb{P} .

We now describe in detail the specific model of random environment that we consider in this paper. Let $(\lambda_k, d_k)_{k \in \mathbb{Z}}$ be a stationary and ergodic sequence of pairs, such that $\lambda_k \in (0, 1)$ and $d_k \in \mathbb{N}$. Throughout the paper we denote by P the joint law of the sequence of pairs $(\lambda_k, d_k)_{k \in \mathbb{Z}}$. For $n \in \mathbb{Z}$ let

$$a_n = \begin{cases} \sum_{k=1}^n d_k & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -\sum_{k=-n}^{-1} d_k & \text{if } n < 0. \end{cases}$$

The random variables a_n serve as locations of random impurities in the, otherwise homogeneous, random medium. More precisely, the *sparse random environment* ω associated with P is defined by

$$\omega_n = \begin{cases} \lambda_k & \text{if } n = a_k \text{ for some } k \in \mathbb{Z}, \\ 1/2 & \text{otherwise.} \end{cases} \quad (1)$$

For future reference, we also define

$$\rho_n = \frac{1 - \omega_n}{\omega_n} \quad \text{and} \quad \xi_n = \frac{1 - \lambda_n}{\lambda_n}, \quad n \in \mathbb{Z}. \quad (2)$$

We refer to the random walk in the environment defined by (1) as a *random walk in a sparse random environment* (RWSRE). The primary focus of this paper is to illuminate the dependence of the basic properties of RWSREs on the distribution of the sequence $(d_n)_{n \in \mathbb{Z}}$, and compare the dynamics of RWSREs to the dynamics of the classical RWRE, which corresponds to the special case $d_0 = 1$ a. s.

In the classical RWRE model, ω is a stationary and ergodic sequence under P [36]. It is known (see [16, 27, 31, 36] and, for instance, [1, 3, 7] and [9, 10, 21] for some recent advances in the recurrent and transient cases, respectively) that asymptotic results for one-dimensional RWREs can usually be stated in terms of certain averages of functions of ρ_0 and explained

by means of typical “landscape features” (such as *traps* and *valleys*) of the *random potential* $(R_n)_{n \in \mathbb{Z}}$, which is associated with the random environment as follows:

$$R_n = \begin{cases} \sum_{k=1}^n \log \rho_k & \text{if } n > 0, \\ -\sum_{k=1}^{|n|} \log \rho_{-k} & \text{if } n < 0. \end{cases} \quad (3)$$

We remark that interpreting a RWRE as a random walk in the random potential (3) serves to provide heuristic explanations to most results about RWREs, including those discussed in this paper.

In our model, the sparse environment ω_n is defined as a simple functional of the *marked point process* $(a_n, \lambda_n)_{n \in \mathbb{Z}}$, and it is in general non-stationary. However, it is well known that if $E_p(d_0) < \infty$, the underlying probability space can be enlarged to include a random variable M , such that the random shift $(a_{n+M}, \lambda_{n+M})_{n \in \mathbb{Z}}$ of the sequence $(a_n, \lambda_n)_{n \in \mathbb{Z}}$ is stationary and ergodic. Therefore, one should expect that if $E_P(d_0) < \infty$, basic zero-one laws, such as recurrence-transience, existence of asymptotic speed, and ballisticity, are similar to the corresponding features of the RWRE associated with the random environment $\omega = (\lambda_n)_{n \in \mathbb{Z}}$.

However, an analogous claim about the similarity of limit theorems for random walks in the environments λ and ω is less obvious. Indeed, the dependence structure associated with the environment plays a crucial role in such theorems, and it is clearly not preserved under the transformation $\lambda \mapsto \omega$. We remark, for instance, that the transformation of an i.i.d. environment yields a stationary and ergodic transformation of a Markov chain. See Sections 2 and 3.3 for a more detailed discussion, and Section 3.5 for the case $E(d_0) = \infty$.

In the continuous setting, a model which is closely related to the RWSREs discussed in this paper is the multi-skewed Brownian motion introduced in [24]. A direct discrete-time analogue of the multi-skewed Brownian motion is a *multi-skewed random walk*, which can be introduced as a quenched variant of our model when $(\lambda_n)_{n \in \mathbb{Z}}$ is a certain deterministic sequence of constants. In accordance to the physical motivation of the model in [24], the author refers to the marked sites (i.e., the elements of \mathcal{A} in the author’s notation) as interfaces, while the long stretches of “regular” sites between interfaces are referred to by the author as layers.

We remark that certain random environments that consist of alternating stretches of sites of two different types, and induce sub-linear growth rates on the underlying random processes, have been considered in [23, 30] and, in a slightly different context, in [4]. The overlap between results and proof methods in this work and in [23, 30] is minimal, and it is discussed in more detail in Section 3.

Somewhat related to our work is the study of [20, 19], where it is shown that an $X_n \sim (\log n)^\alpha$ asymptotic behavior of the random walk can occur under a perturbation of an i.i.d. recurrent environment $(\omega_n)_{n \in \mathbb{Z}}$ of the form $\omega_n^{\text{new}} = \omega_n + f_n$, where $E_P(\log \frac{1-\omega_0}{\omega_0}) = 0$ and f_n converges to zero in probability as $|n| \rightarrow \infty$. A heuristic explanation of this phenomenon can be provided by considering that both recurrent and transient random walks “spread out,” and hence, for a large class of perturbations f_n , a typical landscape of the environment as viewed from the particle at time n can be identified. Moreover, it can be shown that this typical landscape is a dominant factor in determining the asymptotic behavior of the random walk as n goes to infinity.

The rest of the paper is organized as follows. In Section 2, we introduce a notion of a dual stationary environment along with the Palm dualities that are used in the proof of our results. In Section 3 we state and discuss our main results for the asymptotic behavior of RWSREs. These include recurrence and transience criteria, asymptotic speed, stable laws, and a Sinai-type result for RWSREs. Finally, in Section 4 we present the proofs of our results.

2 Dual stationary environment

The sparse environment introduced in Section 1 is in general a non-stationary sequence. The aim of this section is to introduce a *dual* stationary environment and relate the properties of the RWSRE to the corresponding properties of the RWRE in the dual environment. If $E_p(d_0) < \infty$, then the underlying probability space can be enlarged to include a non-negative random variable M , such that the random shift $(\omega_{n-M})_{n \in \mathbb{Z}}$ of the environment $(\omega_n)_{n \in \mathbb{Z}}$ is stationary and ergodic. Furthermore, the distribution of the sparse environment turns out to be the distribution of its stationary version conditioned on the event $0 \in \mathcal{A}$.

In contrast to the usual RWRE, $\omega = (\omega_n)_{n \in \mathbb{Z}}$ is in general a non-stationary sequence in the RWSRE model. In fact, ω is *cycle-stationary* under \mathbb{P} , namely

$$\theta^{d_n} \omega =_D \omega \quad \text{under } \mathbb{P} \quad \text{for all } n \in \mathbb{Z},$$

where $X =_D Y$ means that the distributions of the random variables X and Y coincide, and the *shift* θ^k is a measurable mapping of (Ω, \mathcal{F}) into itself which is defined for any (possibly random) $k \in \mathbb{Z}$ by

$$(\theta^k \omega)_n = \omega_{n+k}, \quad n \in \mathbb{Z}.$$

If $E_P(d_1) < \infty$ one can define a “stationary dual” $\tilde{\omega}$ of the environment ω as follows [34, 35]. Without loss of generality, we assume that the underlying probability space supports a random variable U , which is independent of ω and is distributed uniformly on the interval $[0, 1] \subset \mathbb{R}$. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the integer part of x , that is $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\}$. We now define $(\tilde{a}_n, \tilde{\omega}_n)_{n \in \mathbb{Z}}$ by setting

$$\tilde{a}_n = a_n + \lfloor U d_0 \rfloor \quad \text{and} \quad \tilde{\omega}_n = \begin{cases} \lambda_k & \text{if } n = \tilde{a}_k \text{ for some } k \in \mathbb{Z}, \\ 1/2 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = (a_n)_{n \in \mathbb{Z}}$ be the set of marked sites of the integer lattice and let $\tilde{\mathcal{A}} = (\tilde{a}_n)_{n \in \mathbb{Z}}$ denote its randomly shifted version introduced above. Furthermore, let

$$e_n = \mathbf{1}_{\{n \in \mathcal{A}\}} \quad \text{and} \quad \tilde{e}_n = \mathbf{1}_{\{n \in \tilde{\mathcal{A}}\}}, \quad n \in \mathbb{Z},$$

and let $\Upsilon := (e_n, \omega_n)_{n \in \mathbb{Z}}$ and $\tilde{\Upsilon} := (\tilde{e}_n, \tilde{\omega}_n)_{n \in \mathbb{Z}}$. Notice that $(\tilde{e}_n, \tilde{\omega}_n)_{n \in \mathbb{Z}} = (\theta^{-M} e_n, \theta^{-M} \omega_n)_{n \in \mathbb{Z}}$, where $M := \lfloor U d_0 \rfloor$.

For sparse environments induced by a renewal sequence a_n , the dual environment can be defined equivalently in a rather explicit manner as a functional of an auxiliary Markov

chain. We will exploit this alternative construction in Section 4.3. The uniqueness of the dual environment (which implies, in particular, that the alternative construction yields the same dual) follows from the reverse “stationary to cycle-stationary” Palm duality described, for instance, in [35, Theorem 1].

The following theorem is an adaptation to our setting of the classical Palm dualities [34, Chapter 8] between the distribution of Υ under P and the distribution of $\tilde{\Upsilon}$ under a measure Q equivalent to P .

Theorem 2.1 (see Theorem 2 in [35]). *Assume that $(\lambda_n, d_n)_{n \in \mathbb{N}}$ is a stationary ergodic sequence under P and $E_P(d_0) < \infty$. Define a new probability measure Q on the Borel subsets of the product set $(\{0, 1\} \times (0, 1))^{\mathbb{Z}}$ by setting*

$$\frac{dQ}{dP(v)} = \frac{d_0(v)}{E_P(d_0)}, \quad v \in (\{0, 1\} \times (0, 1))^{\mathbb{Z}}.$$

Then:

- (a) $(\tilde{e}_n, \tilde{\omega}_n)_{n \in \mathbb{Z}}$ is a stationary and ergodic sequence under Q .
- (b) $P(\mathcal{A} \in \cdot) = Q(\tilde{\mathcal{A}} \in \cdot \mid 0 \in \tilde{\mathcal{A}})$.

We remark that although the claim that the sequence $\tilde{\Upsilon}_n = (\tilde{e}_n, \tilde{\omega}_n)$ is ergodic is not explicitly made in [35], it can be deduced, for instance, from the result of Exercise 1 in [22, p. 56]. The following is a straightforward corollary to Theorem 2.1. For the sake of completeness, the proof is given in the Appendix.

Corollary 2.2. *Under the conditions of Theorem 2.1, we have:*

- (a) $E_P(d_0^2) = E_P(d_0) \cdot E_Q(d_0)$.
- (b) $E_Q(d_0) = 2E_Q(a_0) + 1$.

We remark that the identity $E_Q(d_0) = 2E_Q(a_0) + 1$ can be thought of as a variation of the “waiting time paradox” of the classical renewal theory [8].

3 Asymptotic behavior of RWSRE

In this section, we state the basic limit theorems that describe the asymptotic behavior of the random walk X_n , while the proofs are provided in Section 4. We first state recurrence and transience criteria for RWSRE.

3.1 Recurrence and transience criteria

Let $\sigma_0 = 0$ and

$$\sigma_n = \inf\{k \in \mathbb{N} : k > \sigma_{n-1} \text{ and } X_k \in \mathcal{A}\}.$$

Thus $(\sigma_n)_{n \in \mathbb{N}}$ consists of the times of successive visits of X_n to the random point set \mathcal{A} . Define a nearest-neighbor random walk $(\mathbb{X}_n)_{n \geq 0}$ on \mathbb{Z} by setting

$$\mathbb{X}_n = k \quad \text{if and only if} \quad X_{\sigma_n} = a_k. \tag{4}$$

Taking into account the solution of the gambler's ruin problem for the simple symmetric random walk, we note that \mathbb{X}_n is a RWRE with quenched transition probabilities given by

$$P_\omega(\mathbb{X}_{n+1} = j | \mathbb{X}_n = i) = \begin{cases} \xi_i \cdot \frac{1}{d_i} & \text{if } j = i + 1 \\ (1 - \xi_i) \cdot \frac{1}{d_{i-1}} & \text{if } j = i - 1 \\ \xi_i \cdot \frac{d_i - 1}{d_i} + (1 - \xi_i) \cdot \frac{d_{i-1} - 1}{d_{i-1}} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Moreover, $\limsup_{n \rightarrow \infty} X_n = \limsup_{n \rightarrow \infty} \mathbb{X}_n$ and $\liminf_{n \rightarrow \infty} X_n = \liminf_{n \rightarrow \infty} \mathbb{X}_n$, \mathbb{P} - a. s. Thus, under very mild conditions, recurrence and transience criteria for the RWSRE X_n can be derived directly from the corresponding criteria for the RWRE \mathbb{X}_n (see, for instance, [36, Theorem 2.1.2] for the latter). More precisely, we have:

Theorem 3.1. *Suppose that the following three conditions are satisfied:*

1. *The sequence of pairs $(d_n, \lambda_n)_{n \in \mathbb{Z}}$ is stationary and ergodic*
2. *$E_P(\log \xi_0)$ exists (possibly infinite)*
3. *$E_P(\log d_0) < \infty$.*

Then:

- (a) *$E_P(\log \xi_0) < 0$ implies $\lim_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} - a. s.*
- (b) *$E_P(\log \xi_0) > 0$, implies $\lim_{n \rightarrow \infty} X_n = -\infty$, \mathbb{P} - a. s.*
- (c) *$E_P(\log \xi_0) = 0$ implies $\liminf_{n \rightarrow \infty} X_n = -\infty$ and $\limsup_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} - a. s.*

Theorem 3.1 implies that as long as $E_P(\log d_0)$ is finite, the sparse environment ω induces the same recurrence-transience behavior as the underlying random environment λ . The following theorem shows that the opposite phenomenon occurs when $E_P(\log d_0) = +\infty$. Namely, the properties of λ are essentially irrelevant to the basic asymptotic behavior of X_n .

Theorem 3.2. *Suppose that the following conditions hold:*

1. *The sequence of pairs $(d_n, \lambda_n)_{n \in \mathbb{Z}}$ is stationary and ergodic*
2. *The random variables d_n are i.i.d.*
3. *$E_P(|\log \xi_0|) < +\infty$ while $E_P(\log d_0) = +\infty$.*

Then, $\liminf_{n \rightarrow \infty} X_n = -\infty$ and $\limsup_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} - a. s.

We remark that $(d_n)_{n \in \mathbb{N}}$ is not necessarily independent of $(\lambda_n)_{n \in \mathbb{Z}}$. The proof of Theorem 3.2 is given in Section 4.1.

3.2 Transient RWSRE: asymptotic speed

We now turn our attention to the law of large numbers for X_n . Whenever it exists, $\lim_{n \rightarrow \infty} X_n/n$ is referred to as the *asymptotic speed* of the random walk. Let $T_0 = 0$ and for $n \in \mathbb{N}$,

$$T_n = \inf\{k \geq 0 : X_k = n\} \quad \text{and} \quad \tau_n = T_{a_n} - T_{a_{n-1}}. \quad (6)$$

Let

$$\bar{S} = 1 + 2 \sum_{i=0}^{\infty} \prod_{j=0}^i \xi_j \quad \text{and} \quad \bar{F} = 1 + 2 \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \xi_{-j}^{-1}. \quad (7)$$

We have the following:

Theorem 3.3. *Let the conditions of Theorem 3.1 hold. Suppose in addition that $(d_n)_{n \in \mathbb{Z}}$ is independent of $(\lambda_n)_{n \in \mathbb{Z}}$ under P . Then the asymptotic speed of the RWSRE exists $\mathbb{P} - \text{a. s.}$ Moreover,*

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} X_n/n = v_P\right) = \mathbb{P}\left(\lim_{n \rightarrow \infty} T_n/n = 1/v_P\right) = 1,$$

where $v_P \in (-1, 1)$ is a constant whose reciprocal v_P^{-1} is equal to

$$\begin{aligned} \frac{1}{v_P} &= \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = +\infty\}} \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(\bar{S}) \cdot E_P(d_0) \right] \\ &\quad - \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = -\infty\}} \cdot \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(\bar{F}) \cdot E_P(d_0) \right], \quad \mathbb{P} - \text{a. s.} \end{aligned} \quad (8)$$

Notice that if λ_i (and hence ξ_i) are i.i.d., then (8) reduces to

$$\begin{aligned} \frac{1}{v_P} &= \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = +\infty\}} \cdot \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(d_0) \cdot \frac{1 + E_P(\xi_0)}{1 - E_P(\xi_0)} \right] \\ &\quad - \mathbf{1}_{\{\lim_{n \rightarrow \infty} X_n = -\infty\}} \cdot \left[\frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(d_0) \cdot \frac{1 + E_P(\xi_0)}{1 - E_P(\xi_0)} \right], \quad \mathbb{P} - \text{a. s.} \end{aligned}$$

In order to compare (8) with the corresponding result for the regular RWRE, note that if $\lim_{n \rightarrow \infty} X_n = +\infty$, $\mathbb{P} - \text{a. s.}$, then (8) reduces to

$$\frac{1}{v_P} = \frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(d_0) \cdot E_P(\bar{S}), \quad \mathbb{P} - \text{a. s.}$$

Recall the dual environment $\tilde{\omega}$ defined in Section 2, and let

$$\tilde{\rho}_n = \frac{1 - \tilde{\omega}_n}{\tilde{\omega}_n}, \quad n \in \mathbb{Z}, \quad \text{and} \quad \tilde{S} = \sum_{i=1}^{\infty} (1 + \tilde{\rho}_{-i}) \prod_{j=0}^{i-1} \tilde{\rho}_{-j} + 1 + \tilde{\rho}_0. \quad (9)$$

It is well known that the asymptotic speed of the usual RWRE is given by $1/E_P(\tilde{S})$ (see, for instance, [36, Theorem 2.1.9]). The proof of the following proposition is straightforward, and it is provided in the Appendix.

Proposition 3.4. *Let the conditions of Theorem 3.3 hold. Suppose in addition that*

1. $\lim_{n \rightarrow \infty} X_n = +\infty$, \mathbb{P} - a. s.
2. $E_P(d_0) < \infty$.

Then, $v_P = 1/E_Q(\tilde{S})$.

We remark that a proposition similar to Proposition 3.4 can be obtained when the random walk is transient to the left (i.e., when $\lim_{n \rightarrow \infty} X_n = -\infty$, \mathbb{P} - a. s.) by replacing $\tilde{\rho}_{-k}$ with ρ_k^{-1} in the formula (9) for \tilde{S} .

Theorem 3.3 immediately yields the following version of Theorem 1.3 and Corollary 1.4 in [23]. For any constants $\mu > 0$ and $\nu \geq 0$, we denote by $\mathcal{P}_{\mu,\nu}^\circ$ the set of distributions $(\lambda_n, d_n)_{n \in \mathbb{Z}}$ for which the conditions of Theorem 3.3 hold and

$$E_P(d_0) = \mu \quad \text{and} \quad 1/E_P(\bar{S}) = \nu.$$

We then have:

Corollary 3.5. $\max_{P \in \mathcal{P}_{\mu,\nu}^\circ} v_P = \nu/\mu$. *Furthermore, the maximum is attained at $P \in \mathcal{P}_{\mu,\nu}^\circ$ if and only if $\text{VAR}_P(d_0) = 0$.*

Combining this result with [33, Theorem 4.1], we obtain the following corollary. For any constants $\mu > 0$ and $b < 0$, we denote by $\mathcal{P}_{\mu,\nu}^*$ the set of distributions $(\lambda_n, d_n)_{n \in \mathbb{Z}}$ for which the conditions of Theorem 3.3 hold and

$$E_P(d_0) = \mu \quad \text{and} \quad E_P(\log \xi_0) = b.$$

Corollary 3.6. *We have:*

$$\max_{P \in \mathcal{P}_{\mu,\nu}^*} v_P = \frac{1}{\mu} \cdot \frac{1 - e^b}{1 + e^b}.$$

Furthermore, the maximum is attained at $P \in \mathcal{P}_{\mu,\nu}^\circ$ if and only if $\text{VAR}_P(d_0) = \text{VAR}_P(\lambda_0) = 0$, in which case $\lambda_0 = \frac{1}{1+e^b}$, P - a. s.

The slowdown of a random walk in a random environment, as compared to a simple random walk, is a well-known general phenomenon [11, 31, 36] that can be explained heuristically by fluctuations in the associated random potential. For example, a random walk transient to the right will quickly pass stretches of the environment that “push” it forward, but will be “trapped” for a long time in atypical stretches that “push” it backward.

3.3 Stable limit laws for transient RWSRE

The aim of this section is to derive non-Gaussian limit laws for transient random walks in a sparse random environment. The existence of the stationary dual environment suggests that the limit theorems can be first obtained for the random walk in the dual environment and then translated into the corresponding results for the RWSRE. In what follows, we adopt this approach even though it has the shortcoming of restricting our derivation to a class of

i.i.d. environments for which stable laws in the dual setting are known. It appears plausible that alternative methodologies, which would be considerably more technically involved, such as a direct generalization of the “branching process” approach of [16, 18], or an adaptation of the “random potential” method developed in [10], would allow to extend the results presented in this chapter to a larger class of i.i.d. environments (and also perhaps to some Markov-dependent environments).

We will adopt here the following set of assumptions:

Assumption 3.7.

- (A1) $(\lambda_n)_{n \in \mathbb{Z}}$ is an i.i.d. sequence
- (A2) $(d_n)_{n \in \mathbb{Z}}$ is an i.i.d. sequence independent of $(\lambda_n)_{n \in \mathbb{Z}}$
- (A3) $P(\epsilon < \lambda_0 < 1 - \epsilon) = 1$ for some $\epsilon \in (0, 1/2)$.
- (A4) For some $\kappa > 0$,

$$E_P(\xi_0^\kappa) = 1 \tag{10}$$

- (A5) There exists a constant $M > 0$ such that $P(d_0 < M) = 1$.
- (A6) The distribution of $\log \xi_0$ is non-arithmetic, that is $P(\log \xi_0 \in \alpha \mathbb{Z}) < 1$ for all $\alpha \in \mathbb{R}$.

Notice that (A4) implies by Jensen’s inequality that $E_P(\log \xi_0) \leq 0$. In view of (A6), the inequality is strict and hence the random walk is transient to the right. We remark that although condition (A5) appears to be required for our proof, it is likely that can be relaxed or even omitted.

For any $\kappa \in (0, 2]$ and $b > 0$, we denote by $\mathcal{L}_{\kappa,b}$ the stable law of index κ with the characteristic function

$$\log \widehat{\mathcal{L}}_{\kappa,b}(t) = -b|t|^\kappa \left(1 + i \frac{t}{|t|} f_\kappa(t) \right), \tag{11}$$

where $f_\kappa(t) = -\tan \frac{\pi}{2} \kappa$ if $\kappa \neq 1$ and $f_1(t) = 2/\pi \log t$. With a slight abuse of notation we use the same symbol for the distribution function of this law. If $\kappa < 1$, $\mathcal{L}_{\kappa,b}$ is supported on the positive reals, and if $\kappa \in (1, 2]$, $\mathcal{L}_{\kappa,b}$ has zero mean [26, Chapter 1]. For $\kappa = 2$, the law $\mathcal{L}_{2,b}$ is a normal distribution with zero mean and variance equal to $2b$.

We have:

Theorem 3.8. *Suppose that Assumption 3.7 is satisfied. Then the following hold for some $b > 0$:*

- (i) If $\kappa \in (0, 1)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-\kappa} X_n \leq \mathfrak{z}) = 1 - \mathcal{L}_{\kappa,b}(\mathfrak{z}^{-1/\kappa})$,
- (ii) If $\kappa = 1$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1}(\log n)^2(X_n - \delta(n)) \leq \mathfrak{z}) = 1 - \mathcal{L}_{1,b}(-\mathfrak{z})$, for suitable $A_1 > 0$ and $\delta(n) \sim (A_1 \log n)^{-1}n$,
- (iii) If $\kappa \in (1, 2)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/\kappa}(X_n - n\nu_P) \leq \mathfrak{z}) = 1 - \mathcal{L}_{\kappa,b}(-\mathfrak{z})$,
- (iv) If $\kappa = 2$, then $\lim_{n \rightarrow \infty} \mathbb{P}((n \log n)^{-1/2}(X_n - n\nu_P) \leq \mathfrak{z}) = \mathcal{L}_{2,b}(\mathfrak{z})$.

One can readily see that in the context of Theorem 3.8, if $\kappa > 2$, then the standard CLT holds (it follows, e.g., from [36, Theorem 2.2.1]).

For the hitting times T_n , we have:

Proposition 3.9. *Let the conditions of Theorem 3.8 hold. Then the following hold for some $\tilde{b} > 0$:*

- (i) *If $\kappa \in (0, 1)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/\kappa} T_n \leq t) = \mathcal{L}_{\kappa, \tilde{b}}(t)$,*
- (ii) *If $\kappa = 1$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1}(T_n - nD(n)) \leq t) = \mathcal{L}_{1, \tilde{b}}(t)$, for suitable $c_0 > 0$ and $D(n) \sim c_0 \log n$,*
- (iii) *If $\kappa \in (1, 2)$, then $\lim_{n \rightarrow \infty} \mathbb{P}(n^{-1/\kappa} (T_n - nv_P^{-1}) \leq t) = \mathcal{L}_{\kappa, \tilde{b}}(t)$,*
- (iv) *If $\kappa = 2$, then $\lim_{n \rightarrow \infty} \mathbb{P}((n \log n)^{-1/2}(T_n - nv_P^{-1}) \leq t) = \mathcal{L}_{2, \tilde{b}}(t)$.*

It can be shown that the value of the parameter b in the statement of Theorem 3.8 is solely determined by the distribution of λ_0 ; in particular, it is independent of the distribution of d_0 , provided that d_0 satisfies the conditions of the theorem. This result may appear surprising at first, especially in view of a large deviation interpretation of κ given in [36, Section 2.4] (it is not hard to see that the rate functions of the random potentials associated with the sequences ξ_n and ρ_n are actually different). However, it can be explained in terms of the associated branching process and the corresponding interpretation for κ . Furthermore, a careful inspection of the proof given in Section 4.3 shows that both parameters b and \bar{b} of the limiting distributions are decreasing functions of $E_P(d_0)$ and increasing functions of $VAR(d_n)$. This can be explained by the fact that b , in some rigorous sense, plays the role of the variance for the stable laws $\mathcal{L}_{\kappa, b}$; see, for instance, the form of the characteristic function in (11) and compare it to the characteristic function of a normal distribution.

3.4 Limit theorems for recurrent RWSRE

The goal of this section is to obtain a generalization of Sinai's limit theorem for a class of recurrent RWSREs. The main result is stated in Theorem 3.11. A suitable normalized random potential for the RWSRE is introduced in Lemma 4.4. The notion of a valley of the random potential, which is essential for understanding the behavior of Sinai's model [27, 36], is directly carried over to our setup. The proof of the main result is presented in Section 4.4.

Sinai [27] studied a recurrent RWRE X_n and showed that

$$\frac{\sigma^2}{(\log n)^2} X_n \Rightarrow b_\infty,$$

where b_∞ is random variable which can be described as the "location of the deepest valley" of a Brownian motion. The proof of Sinai [27] uses a construction that implements the idea that a properly scaled recurrent RWRE can be thought of as the motion of a particle in a suitably normalized random potential W_n . The normalized potential converges to a Brownian motion, and Sinai's result shows a remarkable slowing down of the diffusive time scale. The density function of the limit distribution b_∞ was characterized independently by Kesten [15]

and Golosov [12, 13], who obtained that

$$P(b_\infty \in dx) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{-\frac{(2k+1)^2\pi^2}{8}|x|\right\} dx$$

In this paper, we derive a limit theorem for a recurrent random walk in a sparse random environment under the following assumption: Let $\alpha \in (0, 1)$ and assume that d_1 is in the domain of attraction of a stable law with index α . Namely,

$$P(d_1 > t) = t^{-\alpha} h(t), \quad t \geq 1, \quad (12)$$

where $h(t)$ is slowly varying at infinity, that is $h(\lambda t) \sim h(t)$ as t goes to $+\infty$ for all $\lambda > 0$. In particular, we define $S_n = n^{-1} \sum_{k=1}^{n-1} \log \xi_k$ and assume the following:

Assumption 3.10.

1. $E_P(\log \xi_0) = 0$ (recurrence)
2. $\sigma_P^2 := E_P(\log^2 \xi_0) \in (0, \infty)$
3. $\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \log \rho_k \Rightarrow B(t)$
4. $P(d_1 > t) = t^{-\alpha} h(t)$, where $\alpha \in (0, 1)$ and $h(t)$ is slowly varying.

Recall that a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is said to be *regularly varying of index* $\alpha \in \mathbb{R}$ if $f(t) = t^\alpha h(t)$ for a slowly varying $h : \mathbb{R}_+ \rightarrow \mathbb{R}$. We denote the set of all regularly varying functions of index α by \mathcal{R}_α . We have the following:

Theorem 3.11. *Let Assumption 3.10 hold and fix any $\delta > 0$. Then, there is a function $u \in \mathcal{R}_{2/\alpha}$ such that the following holds: For any $\varepsilon > 0$ and $\delta \in (0, 1)$, there is an integer n_1 such that for all $n > n_1$ there exist a set of environments $C_n \subset \Omega$ and a random variable $b_n = b_n(\omega)$ such that $P(C_n) \geq 1 - \delta$ and*

$$\lim_{n \rightarrow \infty} P_\omega \left(\left| \frac{X_n}{u(\log n)} - b_n \right| > \varepsilon \right) = 0$$

uniformly in $\omega \in C_n$. Moreover, as $n \rightarrow \infty$ the probability distribution for b_n converges weakly to a non-degenerate limiting distribution b_∞ .

We remark that Sineva [28, 30] obtained similar limit laws for different variations of Sinai's model. In all these results, the limiting distribution of a properly scaled random walk X_n admits a representation as the deepest valley of an auxiliary process, which in turn is obtained as the weak limit of a suitably defined random potential. For a definition of the notion of a valley in this context, we refer the reader to [27] or [36].

3.5 Environment viewed from the position of the particle

In this section we study the “environment viewed from the particle” process $(\theta^{X_n} \omega)_{n \geq 0}$ for a transient RWSRE. It is not hard to see that the pair $(\theta^{X_n} \omega, X_n)$ forms a Markov chain, which allows to consider X_n as a functional (projection into the second coordinate) of a

Markov process. Even though the state space of this Markov chain is considerably large, the representation is useful due to the fact the underlying Markov chain turns out to be stationary and ergodic in the transient regime. The concept of the environment viewed from the particle was introduced by S. Kozlov in a broader context in [17] (see also [31] and [5, 6, 32]). In Section 3.2, we proved the existence of the asymptotic speed $v_P := \lim_{n \rightarrow \infty} X_n$ for RWSREs associated with a stationary and ergodic environment $(d_n, \lambda_n)_{n \in \mathbb{Z}}$ by using a direct approach. In fact, using the techniques described in [36, Section 2.2] and the existence of the dual environment, one can prove the following result. Similarly to (2), let

$$\xi_n = \frac{1 - \lambda_n}{\lambda_n}, \quad n \in \mathbb{Z}. \quad (13)$$

We have:

Theorem 3.12. *Consider a random walk X_n in a stationary and ergodic sparse environment $(\lambda_n, d_n)_{n \in \mathbb{Z}}$. Assume that $E_P(\log \xi_0)$ is well defined (possibly infinite) and $E_P(d_0) < \infty$. Then*

- (a) $v_P > 0$ if and only if there exists a stationary distribution P° equivalent to P for the Markov chain $\bar{\omega}_n = \theta^{X_n} \omega$, $n \geq 0$. If such a distribution P° exists it is unique and is given by the following formula:

$$P^\circ(B) = v_P E_Q \left[E_\omega^0 \left(\sum_{n=0}^{T_1-1} \mathbf{1}_{\{\bar{\omega}_n \in B\}} \right) \right], \quad B \in \mathcal{F}, \quad (14)$$

where $\bar{\omega}_n := \theta^{\xi_n} \omega$ and Q is the distribution of the dual environment.

- (b) $(\bar{\omega}_n)_{n \geq 0}$ is an erodic process under $\mathbb{P}^\circ := P^\circ \otimes P_\omega$.

- (c) $\frac{dP^\circ}{dP} = \frac{dQ}{dP} \times \Lambda(\omega) = \frac{d_0 \cdot \Lambda(\omega)}{E_P(d_0)}$, where

$$\Lambda(\omega) := \frac{1}{\omega_0} \left[1 + \sum_{i=1}^{\infty} \prod_{j=1}^i \rho_j \right] = \frac{1}{\omega_0} \left[d_1 + \sum_{i=1}^{\infty} d_{i+1} \times \prod_{j=1}^i \xi_j \right]. \quad (15)$$

- (d) $v_P = 1/E_Q(\Lambda) = \frac{E_P(d_0)}{E_P(d_0 \Lambda)}$.

With one exception, the proof of Theorem 3.12 follows along the lines of the corresponding results in [36, Section 2.1] (namely, Lemmas 2.1.18, 2.1.20, 2.1.25, and Corollary 2.1.25 therein). The only exception is the proof that the existence of the environment viewed from the position of the particle actually implies $v_P > 0$. The latter can be obtained by a straightforward modification of the proofs of [5, Theorem 3.5 (ii)] or [25, Theorem 2.3] for instance. The proof of Theorem 3.12 is therefore omitted.

Remark 3.13. *The asymptotic speed for the simple nearest-neighborhood random walk on \mathbb{Z} with probability of jumps forward p and jumps backward $q = 1 - p$ is $(p - q) = 2p - 1$. Although v_P is not equal to $E_Q(2\omega_0 - 1)$, quite remarkably it turns out to be equivalent to $E_{P^\circ}(2\omega_0 - 1)$ (compare, for instance, with formula (2.1.29) in [36]).*

4 Proofs

4.1 Proof of Theorem 3.2

Recall the definitions of ρ_n and ξ_n in (2). Furthermore, for a non-zero integer n let η_n be the number of marked sites within the closed interval $I_n = \text{sign}(n) \cdot [1, n]$. More precisely, let $\eta_0 = 0$ and

$$\eta_n = \chi(I_n \cap \mathcal{A}) = \begin{cases} \sum_{k=1}^n \mathbf{1}_{\{k \in \mathcal{A}\}} & \text{if } n > 0 \\ \sum_{k=1}^n \mathbf{1}_{\{-k \in \mathcal{A}\}} & \text{if } n < 0. \end{cases} \quad (16)$$

Notice that by the ergodic theorem,

$$\lim_{|n| \rightarrow \infty} \frac{\eta_n}{|n|} = Q(0 \in \mathcal{A}) = \frac{1}{E_P(d_1)}, \quad P - \text{a.s. and } Q - \text{a.s.} \quad (17)$$

Denote

$$S(\omega) = \sum_{k=1}^{\infty} \rho_1 \rho_2 \cdots \rho_k \quad \text{and} \quad F(\omega) = \sum_{k=0}^{\infty} \rho_0^{-1} \rho_{-1}^{-1} \cdots \rho_{-k}^{-1}.$$

To prove Theorem 3.2 it suffices (see, for instance, the proof of [36, Theorem 2.1.2]) to show that the conditions of the theorem imply

$$P(S(\omega) = F(\omega) = +\infty) = 1 \quad (18)$$

Toward this end, note that $\eta_{a_n} = \eta_{a_{n+1}} = \dots = \eta_{a_{n+1}-1} = n$ for $n \geq 0$, and hence

$$S(\omega) = \sum_{k=1}^{\infty} \xi_1 \xi_2 \cdots \xi_{\eta_k} = (a_1 - 1) + \sum_{n=1}^{\infty} \xi_1 \xi_2 \cdots \xi_n \cdot d_{n+1}, \quad (19)$$

where, to claim the first identity, we used the standard convention that $\xi_1 \xi_2 \cdots \xi_{\eta_k} = 1$ if $\eta_k = 0$. Similarly, $\eta_{a_n} = \eta_{a_{n+1}} = \dots = \eta_{a_{n+1}-1} = n + 1$ for $n < 0$, and hence

$$F(\omega) = \sum_{k=1}^{\infty} \xi_0^{-1} \xi_{-1}^{-1} \xi_{-2}^{-1} \cdots \xi_{-\eta_k}^{-1} = \sum_{n=0}^{\infty} \xi_0^{-1} \xi_{-1}^{-1} \cdots \xi_{-n}^{-1} \cdot d_{-n}. \quad (20)$$

Furthermore, the condition $E_P(\log d_0) = +\infty$ implies that $\sum_{n=1}^{\infty} P(\log d_0 > M \cdot n) = \infty$ for any $M > 0$. Thus, since d_n are i.i.d., it follows from the second Borel-Cantelli lemma that $P(\log d_n > M \cdot n \text{ i.o.}) = 1$ for any $M > 0$. Hence, the ergodic theorem along with the condition $E_P(|\log \xi_0|) < +\infty$ imply that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \left(d_{n+1} \cdot \prod_{k=1}^n \xi_k \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{k=1}^n \log \xi_k + \log d_{n+1} \right) = +\infty, \quad P - \text{a.s.},$$

which yields $P(S(\omega) = +\infty) = 1$. A similar argument shows that, under the conditions of the theorem, $P(F(\omega) = +\infty) = 1$ and hence (18) holds, as desired. \square

4.2 Proof of Theorem 3.3

The proof is an adaption of the corresponding arguments for the regular RWRE. See, for instance, [36, Section 2.2].

Recall the definitions of T_n and τ_n in (16). We have:

Lemma 4.1. *Assume that the conditions of Theorem 3.1 hold and suppose, in addition, that $\mathbb{P}(\limsup_{n \rightarrow \infty} X_n = 1)$. Then $(\tau_n)_{n \in \mathbb{N}}$ is a stationary and ergodic sequence under the law \mathbb{P} .*

Proof of Lemma 4.1. Enlarge, if needed, the underlying probability space to include a sequence of i.i.d. random variables $\gamma = (\gamma_{x,n})_{x \in \mathbb{Z}, n \in \mathbb{N}}$ such that

1. γ is independent of ω under the law P , and
2. each random variable $\gamma_{x,n}$ is distributed uniformly on the interval $[0, 1]$.

Let \mathbb{N}_0 denote the set of non-negative integers, that is $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $x \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, let $l_x(n) = \sum_{t=0}^n \mathbf{1}_{\{X_t=x\}}$ be the number of visits of the random walk to the site x by the time n . For $n \in \mathbb{N}_0$, denote $l_n = l_{X_n}(n)$, $\gamma_n = \gamma_{X_n, l_n}$, and $\bar{\omega}_n = \omega_{X_n}$. Without loss of generality, we can assume that X_n is defined recursively as follows:

$$X_{n+1} = X_n + \mathbf{1}_{\{\gamma_n < \bar{\omega}_n\}} - \mathbf{1}_{\{\gamma_n \geq \bar{\omega}_n\}}.$$

For $n \in \mathbb{N}$, let $C_n = (\xi_x, d_j, \Gamma_x)_{j,x \leq n}$, where $\Gamma_x = (\gamma_{k,i})_{k \leq x, i \in \mathbb{N}_0}$. The sequence $(\tau_n)_{n \in \mathbb{N}}$ defined by (6) is stationary under \mathbb{P} because $(\lambda_n, d_n)_{n \in \mathbb{Z}}$ is stationary and $\mathbb{P}(T_n < \infty) = 1$ for all $n > 0$ under the conditions of the lemma. Furthermore, in the enlarged probability space $(\tau_n)_{n \in \mathbb{N}}$ becomes a deterministic function of $(C_k)_{k \leq n}$. This completes the proof of the lemma since the sequence $(C_n)_{n \in \mathbb{N}}$ is stationary and ergodic under P in the enlarged probability space. \square

Under the conditions of Lemma 4.1, the ergodic theorem yields

$$\frac{T_{a_n}}{n} = \frac{1}{n} \sum_{i=1}^n \tau_{a_i} \rightarrow \mathbb{E}(\tau_{a_1}) \quad \text{as } n \rightarrow \infty, \quad \mathbb{P} - \text{a. s.}$$

We have

Lemma 4.2. *Assume that the conditions of Theorem 3.1 hold and suppose, in addition, that λ and \mathcal{A} are independent under P . Then:*

- (a) $\mathbb{E}(T_{a_1}) = \text{VAR}_P(d_1) + E_P(\bar{S}) \cdot [E_P(d_1)]^2$,
- (b) $\mathbb{E}(T_{a_{-1}}) = \text{VAR}_P(d_1) + E_P(\bar{F}) \cdot [E_P(d_1)]^2$.

Proof of Lemma 4.2. We will only prove the result in (a), the proof of (b) being similar. To evaluate T_{a_1} , we will use a decomposition of the paths of the random walk according to its first step:

$$\begin{aligned} T_{a_1} = & 1 + \mathbf{1}_{\{X_1=1\}} [\mathbf{1}_{\{\tilde{T}_0 < \tilde{t}_{a_1}\}} (\tilde{T}_0 + T'_{a_1}) + \mathbf{1}_{\{\tilde{T}_0 > \tilde{t}_{a_1}\}} \tilde{T}_{a_1}] \\ & + \mathbf{1}_{\{X_1=-1\}} [\mathbf{1}_{\{\hat{T}_0 < \hat{t}_{a_{-1}}\}} (\hat{T}_0 + T''_{a_1}) + \mathbf{1}_{\{\hat{T}_0 > \hat{t}_{a_{-1}}\}} (\hat{T}_{a_{-1}} + T'_0 + T'''_{a_1})], \end{aligned} \tag{21}$$

where

$$\begin{aligned}
\widetilde{T}_0 &= \inf\{n > T_1 : X_n = 0\}, & \widetilde{T}_0 + T'_{a_1} &= \inf\{n > \widetilde{T}_0 : X_n = a_1\}, \\
\widetilde{T}_{a_1} &= \inf\{n > T_1 : X_n = a_1\}, & \widehat{T}_0 &= \inf\{n > T_{-1} : X_n = 0\}, \\
\widehat{T}_0 + T''_{a_1} &= \inf\{n > \widehat{T}_0 : X_n = a_1\}, & \widehat{T}_{a_{-1}} &= \inf\{n > T_{-1} : X_n = a_{-1}\}, \\
\widehat{T}_{a_{-1}} + T'_0 &= \inf\{n > \widehat{T}_{a_{-1}} : X_n = 0\}, & \widehat{T}_{a_{-1}} + T'_0 + T'''_{a_1} &= \inf\{n > T_{a_{-1}} + T'_0 : X_n = a_1\}.
\end{aligned}$$

Taking quenched expectations $E_\omega(\cdot)$ in both sides of (6) yields

$$\begin{aligned}
E_\omega(T_{a_1}) &= 1 + \lambda_0[\mathcal{E}_1(T_0 \wedge T_{a_1}) + \mathcal{P}_1(T_0 < T_{a_1})E_\omega(T_{a_1})] \\
&\quad + (1 - \lambda_0)[\mathcal{E}_{-1}(T_0 \wedge T_{a_{-1}}) + E_\omega(T_{a_1}) + \mathcal{P}_{-1}(T_{a_{-1}} < T_0)E_{\omega, a_{-1}}(T_0)].
\end{aligned}$$

Using the solution of the gambler's ruin problem under \mathcal{P} , we obtain

$$\begin{aligned}
\frac{\lambda_0}{a_1}E_\omega(T_{a_1}) &= 1 + \lambda_0(a_1 - 1) + (1 - \lambda_0)(|a_{-1}| - 1) + \frac{1 - \lambda_0}{|a_{-1}|}E_{\omega, a_{-1}}(T_0) \\
&= \lambda_0 a_1 + (1 - \lambda_0)|a_{-1}| + \frac{1 - \lambda_0}{|a_{-1}|}E_{\omega, a_{-1}}(T_0).
\end{aligned}$$

Thus,

$$\frac{1}{a_1}E_\omega(T_{a_1}) = a_1 + \xi_0|a_{-1}| + \xi_0 \cdot \frac{1}{|a_{-1}|}E_{\omega, a_{-1}}(T_0).$$

Iterating yields

$$\frac{1}{a_1}E_\omega(T_{a_1}) = a_1 + 2 \sum_{k=0}^{\infty} \xi_0 \xi_{-1} \cdots \xi_{-k} \cdot d_{-k}.$$

Taking expectations with respect to P , and using a truncation argument similar to that given in the proof of [36, Lemma 2.1.12] in order to verify when $\mathbb{E}(T_{a_1}) < \infty$, we obtain

$$\mathbb{E}(T_{a_1}) = \text{VAR}_P(d_1) + [E_P(d_1)]^2 \cdot E_P(\bar{S}),$$

as desired. This completes the proof of (a) of the lemma. Part (b) can be derived along the same lines, and hence its proof is omitted. \square

In view of Lemma 4.2, we are now in a position to finish the proof of Theorem 3.3. Variations of Lemma 4.3 below have appeared in a number of references in the field of random walks in random environments. Nonetheless, we provide a proof for the reader's convenience.

Lemma 4.3. *Assume that the conditions of Theorem 3.1 hold and suppose, in addition, that $\lim_{n \rightarrow \infty} \frac{T_{a_n}}{n} = \alpha$, \mathbb{P} -a.s., for some constant $\alpha \leq \infty$. Then,*

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \frac{\alpha}{E_P(d_1)} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{E_P(d_1)}{\alpha}, \quad \mathbb{P} - \text{a. s.}$$

Proof of Lemma 4.3. First, observe that (16) implies

$$a_{\eta_n} \leq n < a_{\eta_n+1}, \quad \mathbb{P} - \text{a. s.}$$

Thus, in view of (17),

$$\lim_{n \rightarrow \infty} \frac{T_n}{n} = \lim_{n \rightarrow \infty} \frac{T_{a_{\eta_n}+1}}{n} = \lim_{n \rightarrow \infty} \frac{T_{a_{\eta_n}}}{\eta_n} \cdot \frac{\eta_n}{n} = \frac{\alpha}{E_P(d_1)}, \quad \mathbb{P} - \text{a. s.}$$

Let now $\zeta(n) \in \mathbb{Z}$ be the unique nonnegative random number such that

$$T_{a_{\zeta(n)}} \leq n < T_{a_{\zeta(n)+1}}. \quad (22)$$

Since X_n is transient to the right, $\mathbb{P}(\lim_{n \rightarrow \infty} \zeta(n) = \infty) = 1$. Furthermore, (22) implies that

$$X_n < a_{\zeta(n)+1} \quad \text{and} \quad X_n \geq a_{\zeta(n)} - (n - T_{a_{\zeta(n)}}).$$

Thus,

$$\frac{a_{\zeta(n)}}{n} - \left(1 - \frac{T_{a_{\zeta(n)}}}{n}\right) \leq \frac{X_n}{n} < \frac{a_{\zeta(n)+1}}{n}.$$

But (22) along with the existence of $\lim_{n \rightarrow \infty} \frac{n}{T_n}$ yield

$$\lim_{n \rightarrow \infty} \frac{a_{\zeta(n)}}{n} = \lim_{n \rightarrow \infty} \frac{a_{\zeta(n)}}{T_{a_{\zeta(n)}}} = \lim_{n \rightarrow \infty} \frac{n}{T_n} = \frac{E_P(d_1)}{\alpha}, \quad \mathbb{P} - \text{a. s.}$$

Hence,

$$\frac{E_P(d_1)}{\alpha} \leq \liminf_{n \rightarrow \infty} \frac{X_n}{n} \leq \limsup_{n \rightarrow \infty} \frac{X_n}{n} \leq \frac{E_P(d_1)}{\alpha},$$

which implies the result in Lemma 4.3. □

The proof of Theorem 3.3 is complete.

4.3 Proof of Theorem 3.8

The proof uses the dual Markovian environment and the reduction to stable limit laws for random walks in a Markovian environment obtained in [18]. Let

$$Y_n = n - a_{\eta_n} = n - \sup\{k \in \mathbb{Z} : k \leq n \text{ and } k \in \mathcal{A}\}, \quad n \in \mathbb{Z}.$$

Notice that $Y_{a_k} = 0$, $k \in \mathbb{N}$, and

$$Y_{n+1} - Y_n = 1 \quad \text{if} \quad a_{\eta_n} \leq n < a_{\eta_n+1}.$$

Let \mathbb{Z}_+ denote the set of non-negative integers, that is $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Observe that if Assumption 3.7 holds then the sequence $Y = (Y_n)_{n \in \mathbb{Z}}$ under the law P is a positive-recurrent Markov chain on \mathbb{Z}_+ whose transition kernel $H(x, y) = P(Y_{n+1} = y | Y_n = x)$ is given by

$$H(x, y) = \begin{cases} \frac{P(d_0 > x+1)}{P(d_0 > x)} & \text{if } y = x + 1, x \in \mathbb{Z}_+ \\ \frac{P(d_0 = x+1)}{P(d_0 > x)} & \text{if } y = 0, x \in \mathbb{Z}_+ \\ 0 & \text{otherwise.} \end{cases}$$

The initial state of Z under P is zero, that is $P(Y_0 = 0) = 1$. It follows then from Theorem 2.1 that Y under the law Q is a stationary Markov chain on \mathbb{Z}_+ whose transition kernel is $H(x, y)$, and the initial distribution is the (unique) invariant distribution of H , that is, using the notation of Theorem 2.1,

$$Q(Y_0 = x) = P([Ud_0] = x) = \frac{P(d_0 > x)}{E_P(d_0)}, \quad x \in \mathbb{Z}_+.$$

It follows that under either P or Q , the sequence (Y, ω) constitutes a two-component Markov chain with transition kernel depending only on the current value of the first component (but not of the second). More precisely, with probability one,

$$\begin{aligned} P(Y_{n+1} = y, \omega_{n+1} \in A | Y_n = x, \omega_n = u) \\ = Q(Y_{n+1} = y, \omega_{n+1} \in A | Y_n = x, \omega_n = u) = \mathbb{H}(x; y, A), \end{aligned}$$

where the stochastic kernel \mathbb{H} on $\mathbb{Z}_+ \times (\mathbb{Z}_+ \times \Omega)$ is given by

$$\mathbb{H}(x; y, A) = H(x, y) \cdot (\mathbf{1}_{\{\frac{1}{2} \in A\}} \cdot \mathbf{1}_{\{y=0\}} + P(\lambda_0 \in A) \cdot \mathbf{1}_{\{y>0\}}), \quad A \in \mathcal{B}([0, 1]).$$

Clearly, the reverse chain $(Y_n)_{n \in \mathbb{Z}}$ is an irreducible Markov chain in the finite state space $\{0, 1, \dots, M-1\}$. In view of the main results of [18], in order to establish that the claims of Theorem 3.8 and Proposition 3.9 hold for the RWSRE X_n and the hitting times T_n , it suffices to verify the following set of conditions for the Markov chain Y_n and the associated Markov hidden model $(Y_{-n}, \lambda_{-n}, d_{-n})_{n \in \mathbb{Z}}$:

$$(B1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log E \left(\prod_{i=0}^{n-1} \rho_{-i}^{\beta_1} \right) \geq 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log E \left(\prod_{i=0}^{n-1} \rho_{-i}^{\beta_2} \right) < 0$$

for some constants $\beta_1 > 0$ and $\beta_2 > 0$.

(B2) The process $q_n = \log \rho_{-n}$ is non-arithmetic relative to (x_n) in the following sense: there do not exist a constant $\alpha > 0$ and a measurable function $\gamma : \mathcal{S} \rightarrow [0, \alpha)$ such that

$$P(q_0 \in \gamma(x_{-1}) - \gamma(x_0) + \alpha \cdot \mathbb{Z}) = 1.$$

Toward this end observe that condition (B1) holds because

$$\begin{aligned} E_P \left(\prod_{i=1}^n \rho_i^\kappa \right) &= E_P \left(\prod_{i=0}^{\eta_n} \xi_i^\kappa \right) = E_P \left(E_P \left(\prod_{i=0}^{\eta_n} \xi_i^\kappa | \eta_n \right) \right) \\ &= E_P \left((E_P(\xi_0^\kappa))^{\eta_n} \right) = E_P(1^{\eta_n}) = 1. \end{aligned}$$

Furthermore, (A6) of Assumption 3.7 along with the fact that ξ_n are i.i.d. trivially implies (B2). The proof is complete.

4.4 Proof of Theorem 3.11

Let $D(\mathbb{R})$ denote the set of real-valued càdlàg functions on $[0, 1]$ equipped with the Skorokhod J_1 -topology. We use the notation \Rightarrow to denote the weak convergence in $D(\mathbb{R})$. We have:

$$U_n(t) := \frac{1}{r_n} \sum_{k=1}^{\lfloor nt \rfloor} d_k \text{ converges weakly to } G_\alpha(t),$$

where $G_\alpha(t)$ is a totally asymmetric stable process with

$$E(e^{i\theta G_\alpha(1)}) = \exp\left\{-|\theta|^\alpha \left(1 - i \operatorname{sign}(\theta) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right\}, \quad \theta \in \mathbb{R}.$$

The key ingredient of the proof is the following (functional) limit theorem for a suitably defined random potential.

Lemma 4.4.

(a) Assume that condition (12) holds with $\alpha \in (0, 1)$. Then, as $n \rightarrow \infty$,

$$\frac{1}{\log n} \sum_{k=1}^{\lfloor u(\log n)t \rfloor} \log \rho_k \Rightarrow V_\alpha,$$

for some sequence $u(n) \in \mathcal{R}_{2\alpha}$.

(b) If $E(d_1) < \infty$, then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \log \rho_k \Rightarrow \mu \sigma_P W,$$

where W is a standard Brownian motion and $V_\alpha = \sigma_P W(G_\alpha^{-1})$.

Proof of Lemma 3.

(a) Let

$$U_n(t) := \frac{1}{r_n} \sum_{k=1}^{\lfloor nt \rfloor} d_k = \frac{1}{r_n} a_{\lfloor nt \rfloor} \quad \text{and} \quad R_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \log \xi_k.$$

It follows from the assumptions of the lemma that, as $n \rightarrow \infty$,

$$U_n(t) \Rightarrow G_\alpha \quad \text{and} \quad R_n(t) \Rightarrow \sigma_P W.$$

for some sequence $r_n \in \mathcal{R}_{\alpha+1}$. Let $U_n^{-1} = n^{-1} \cdot \eta(\lfloor tr_n \rfloor)$ and G_α^{-1} be the inverses in $D(\mathbb{R}_+, \mathbb{R})$ of U_n and G_α , respectively. Then the convergence of U_n and R_n , along with their independence of each other, imply (see, for instance, the derivation of the formula (2.29) in [14]) that in $D(\mathbb{R}_+, \mathbb{R})$,

$$\left(R_n(t), \frac{1}{n} \eta(\lfloor tr_n \rfloor)\right) \Rightarrow (\sigma_P W, G_\alpha^{-1}), \quad \text{as } n \rightarrow \infty.$$

Since the paths of the Brownian motion are continuous, it follows from a random change lemma in [2, p. 151] that $R_n(\frac{1}{n}\eta(\lfloor tr_n \rfloor)) \Rightarrow \sigma_P W(G_\alpha^{-1}(t))$ in $D(\mathbb{R}_+, \mathbb{R})$. That is,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\eta(\lfloor r_n t \rfloor)} \log \xi_k \Rightarrow \sigma_P W(G_\alpha^{-1}), \quad \text{as } n \rightarrow \infty,$$

and, passing to the subsequence $n_k = \log^2 k$, $k \in \mathbb{N}$, we obtain

$$\frac{1}{\log k} \sum_{i=1}^{\eta(\lfloor r_{\log^2 k} t \rfloor)} \log \xi_i = \frac{1}{\log k} \sum_{i=1}^{\lfloor r_{\log^2 k} t \rfloor} \log \rho_i \Rightarrow \sigma_P W(G_\alpha^{-1}), \quad \text{as } k \rightarrow \infty.$$

To conclude the proof of part (a), notice that $r_n \in \mathcal{R}_\alpha$ implies that $r_{\log^2 n} = u(\log n)$ for some sequence $u(n) \in \mathcal{R}_{2\alpha}$, as desired.

(b) We now turn to the proof of part (b) of the theorem. Since $E_p(d_1) < \infty$, the renewal theorem implies

$$\frac{\eta_n}{n} \rightarrow C = \frac{1}{E_p(d_1)}.$$

Therefore (see, for instance, Theorem 14.4 in [2, p. 152]),

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \log \rho_k = \frac{1}{\sqrt{n}} \sum_{k=1}^{\eta(\lfloor nt \rfloor)} \log \xi_k = \frac{\sqrt{\eta(\lfloor nt \rfloor)}}{\sqrt{n}} \frac{1}{\sqrt{\eta(\lfloor nt \rfloor)}} \sum_{k=1}^{\eta(\lfloor nt \rfloor)} \log \xi_k \Rightarrow \mu \sigma_P B(\cdot),$$

where the convergence is the weak convergence in the Skorohod space $D(\mathbb{R}_+)$. \square

Define the normalized random potential associated with the sparse environment as the following:

$$\widehat{R}_n(t) = \text{sign}(t) \cdot \frac{1}{\log n} \sum_{k=1}^{\lfloor u(\log n)t \rfloor} \log \rho_k = \text{sign}(t) \cdot \frac{1}{\log n} \sum_{k=1}^{\eta[\lfloor u(\log n)t \rfloor]} \log \xi_k. \quad (23)$$

By Lemma 3, $\{\widehat{R}_n(t) : t \geq 0\}$ converges weakly in $D(\mathbb{R}_+, \mathbb{R})$ to the process V_α . One can now proceed as in [28] in order to establish Theorem 3.11. In fact, the proof of the main result in [28] is an adaptation of the original argument of Sinai [27] to a situation where the random potential \widehat{R}_n in the form given by (23) converges weakly to a non-degenerate process in $D(\mathbb{R}_+, \mathbb{R})$. We remark that a somewhat shorter derivation of Theorem 3.11 can be obtained by an adaptation of a version of Sinai's argument given in [36, Section 2.5]. The version of the proof in this paper follows the approach of [12] and is due to Dembo, Guionnet, and Zeitouni.

5 Appendix

5.1 Proof of Corollary 2.2

(a) By part (b) of Theorem 2.1,

$$E_Q(d_0) = \int d_0 \cdot \frac{d_0}{E_P(d_0)} dP = \frac{E_P(d_0^2)}{E_P(d_0)},$$

which is equivalent to the the first claim in the corollary.

(b) By definition, $\tilde{a}_0 = \lfloor U d_0 \rfloor$, where U is independent of d_0 under Q . We therefore have,

$$E_Q(\tilde{a}_0) = E_Q(E_Q(\tilde{a}_0|d_0)) = E_Q\left(\frac{1}{d_0} \sum_{k=0}^{d_0-1} j\right) = \frac{1}{2}(E_Q(d_0) - 1),$$

which verifies the second claim in the corollary. \square

5.2 Proof of Proposition 3.4

Recall \bar{S} from (7) and \tilde{S} from (9). First, notice that

$$\tilde{S} = \sum_{i=1}^{\infty} \prod_{j=0}^{i-1} \tilde{\rho}_{-j} + \sum_{i=1}^{\infty} \prod_{j=0}^i \tilde{\rho}_{-j} + 1 + \tilde{\rho}_0 = 1 + 2 \sum_{i=0}^{\infty} \prod_{j=0}^i \tilde{\rho}_{-j}.$$

Next, observe that the following version of the identity (19) can be stated in terms of the “tilde environment” $(\tilde{a}_n, \lambda_n)_{n \in \mathbb{Z}}$:

$$\sum_{n=0}^{\infty} \tilde{\rho}_0 \tilde{\rho}_1 \cdots \tilde{\rho}_n = \tilde{a}_0 + \sum_{n=0}^{\infty} \xi_0 \xi_1 \cdots \xi_n \cdot d_{n+1}. \quad (24)$$

Therefore, using the stationarity of the environment under Q along with the identity (24) and part (b) of Corollary 2.2,

$$\begin{aligned} E_Q(\tilde{S}) &= 1 + 2E_Q\left(\sum_{i=0}^{\infty} \prod_{j=0}^i \tilde{\rho}_i\right) = E_Q(2\tilde{a}_0 + 1) + E_P(d_0) \cdot [E_Q(\bar{S}) - 1] \\ &= E_Q(d_0) + E_P(d_0) \cdot [E_Q(\bar{S}) - 1] = \frac{E_P(d_0^2)}{E_P(d_0)} + E_P(d_0) \cdot [E_Q(\bar{S}) - 1] \\ &= \frac{\text{VAR}_P(d_0)}{E_P(d_0)} + E_P(d_0) \cdot E_P(\bar{S}), \end{aligned}$$

which together with Theorem 3.1 imply that the asymptotic speed of the RWSRE on the event $\{\lim_{n \rightarrow \infty} X_n = +\infty\}$ is $1/E_Q(\tilde{S})$, \mathbb{P} - a. s., under the conditions of Theorem 3.3. \square

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