Appendix A: Mathematical details

Much of the material presented in this appendix is standard knowledge in linear systems theory with elementary extensions and specifications. However, we found it useful to the readers who may not be familiar with this area to include it.

A.1 Linearizing the model

A system dynamics model takes the form of a set of ordinary differential equations,

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)), \tag{A.1}$$

where $\mathbf{x}$ is a column vector of n state variables, $\dot{\mathbf{x}}$ is the vector of first time derivatives (rates), which is a function $\mathbf{f}$ of the state variables and a set of m exogenous or input variables $\mathbf{u}$, and t is the simulated time.[1]

In order to use the concepts of eigenvalue and eigenvector analysis, we linearize the model around a point in time and state space, $\mathbf{x_0} = \mathbf{x}(t_0)$ and $\mathbf{u}_0 = \mathbf{u}(t_0)$, i.e.,

$$\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \approx \mathbf{f}(\mathbf{x_0}, \mathbf{u_0}) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x_0}, \mathbf{u_0})(\mathbf{x}(t) - \mathbf{x_0}) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x_0}, \mathbf{u_0})(\mathbf{u}(t) - \mathbf{u_0}). \tag{A.2}$$

We assume, without loss of generality, that $t_0 = 0$ and define

$$\mathbf{G} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x_0}, \mathbf{u_0}) = \left\{ \frac{\partial \dot{x}_i}{\partial x_j}(\mathbf{x_0}, \mathbf{u_0}) \right\}, \ i, j = 1, \dots, n, \tag{A.3}$$

$$\mathbf{B} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{x_0}, \mathbf{u_0}) = \left\{ \frac{\partial \dot{x}_i}{\partial u_j}(\mathbf{x_0}, \mathbf{u_0}) \right\}, \ i = 1, \dots, n, j = 1, \dots, m, \text{ and} \tag{A.4}$$

$$\mathbf{b} = f(\mathbf{x_0}, \mathbf{u_0}) = \{\dot{x}_i(\mathbf{x_0}, \mathbf{u_0})\}, \ i = 1, \dots, n. \tag{A.5}$$

The matrix **G** is called the *Jacobian* or gain matrix of the system. Substituting the approximation (A.2-A.5) into the original model (A.1) and changing notation to

$$\mathbf{x}(t) - \mathbf{x_0} \rightarrow \mathbf{x}(t),$$
$$\mathbf{u}(t) - \mathbf{u_0} \rightarrow \mathbf{u}(t),$$

leads to the linear system

[1] We henceforth use bold symbols to represent vectors and matrices and non-bold symbols to represent scalar variables. We also use the dot notation to represent time derivatives. An alternative representation of (A.1) uses time as an explicit argument to the rate equations **f**. However, this form can always be converted to (A.1) by an appropriate definition of exogenous variables **u**.

$$\dot{\mathbf{x}}(t) = \mathbf{G}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{b},$$
$$\mathbf{x}(0) = 0. \tag{A.6}$$

In the following, we focus on the significance of the endogenous dynamics arising from interactions of the state variables and therefore assume that the input variables are constant, i.e., $\mathbf{u} = \mathbf{0}$. This leads to the simplified system

$$\dot{\mathbf{x}}(t) = \mathbf{G}\mathbf{x}(t) + \mathbf{b},$$
$$\mathbf{x}(0) = 0. \tag{A.7}$$

A.2 Solution to the linearized model

Differentiating equation (A.7) with respect to time yields the differential equation

$$\ddot{\mathbf{x}}(t) = \mathbf{G}\dot{\mathbf{x}}(t),$$
$$\dot{\mathbf{x}}(0) = \mathbf{b}, \tag{A.8}$$

which can be solved by considering the eigenvalues λ_k and associated right eigenvectors $\mathbf{r_k}$, of **G**, defined as the non-zero solutions to the condition[2]

$$\mathbf{G}\mathbf{r_k} = \lambda_k \mathbf{r_k}. \tag{A.9}$$

Usually, **G** has *n* distinct eigenvalues, in which case the associated eigenvectors will be linearly independent (Chen, 1970, p. 37). (We treat the case of non-distinct eigenvalues in a subsequent section.) Since the eigenvectors form a basis for the space R^n of the state variables, we may express $\dot{\mathbf{x}}(t)$ as a time-varying linear combination $m(t)$ of the eigenvectors,

$$\dot{\mathbf{x}}(t) = m_1(t)\mathbf{r_1} + \cdots + m_n(t)\mathbf{r_n}. \tag{A.10}$$

Differentiating (A.10) with respect to time and comparing it to (A.9), we obtain

[2] When we use the term "eigenvectors" we refer to the right eigenvectors, defined by (A.10) unless otherwise stated. The corresponding left eigenvectors, defined by $\mathbf{l}_k\mathbf{G} = \lambda_k \mathbf{l}_k$, can be found by the relation $[\mathbf{l}_1 \cdots \mathbf{l}_n][\mathbf{r}_1 \cdots \mathbf{r}_n]' = \mathbf{I}$.

$$\begin{aligned}\ddot{\mathbf{x}}(t) &= \dot{m}_1(t)\mathbf{r_1} + \cdots + \dot{m}_n(t)\mathbf{r_n} \\ &= \mathbf{G}\left(m_1(t)\mathbf{r_1} + \cdots m_n(t)\mathbf{r_n}\right) \\ &= m_1(t)\mathbf{G}\mathbf{r}_1 + \cdots m_n(t)\mathbf{G}\mathbf{r_n} \\ &= \lambda_1 m_1(t)\mathbf{r_1} + \cdots \lambda_n m_n(t)\mathbf{r_n},\end{aligned} \tag{A.11}$$

where we have used (A.9) to obtain the last equality. In order for (A.11) to hold true at all times *t*, we must therefore require that

$$\dot{m}_k(t) = \lambda_k m_k(t),\ k = 1,\ldots,n, \tag{A.12}$$

which has the solution

$$m_k(t) = m_k(0)e^{\lambda_k t},\ k = 1,\ldots,n, \tag{A.13}$$

where $m_k(0)$ is a constant. The solution to (A.8) can then be written as

$$\dot{\mathbf{x}}(t) = e^{\lambda_1 t} m_1(0)\mathbf{r}_1 + \ldots + e^{\lambda_n t} m_n(0)\mathbf{r_n}, \tag{A.14}$$

where $e^{\lambda_k t}$ is the *behavior mode* corresponding to the eigenvalue λ_k. The factors $m_k(0)\mathbf{r_k}$ may be found from the initial condition in (A.8),

$$\dot{\mathbf{x}}(0) = m_1(0)\mathbf{r_1} + \cdots + m_n(0)\mathbf{r_n} = \mathbf{b}, \tag{A.15}$$

which, for a given set of eigenvectors $\mathbf{r}$ has a unique solution in *m*.[3] Integrating (A.14) with respect to time yields the solution

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t \dot{\mathbf{x}}(\tau)d\tau \\ &= \left(\frac{m_1(0)}{\lambda_1}\mathbf{r_1}\right)\left(e^{\lambda_1 t} - 1\right) + \cdots + \left(\frac{m_n(0)}{\lambda_n}\mathbf{r_n}\right)\left(e^{\lambda_n t} - 1\right), \\ &= \mathbf{w_0} + \mathbf{w_1}e^{\lambda_1 t} + \ldots + \mathbf{w_n}e^{\lambda_n t}\end{aligned} \tag{A.16}$$

where the weights $\mathbf{w_k}$ are determined by

$$\begin{aligned}\mathbf{w_0} &= -\mathbf{w_1} - \ldots - \mathbf{w_n}, \\ \mathbf{w_k} &= \frac{m_k(0)\mathbf{r_k}}{\lambda_k},\ k = 1,\ldots,n.\end{aligned} \tag{A.17}$$

Here, we have assumed that the eigenvalues are unique and non-zero. We treat these special

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[3] Note that eigenvectors are not uniquely determined since, if a vector is a solution to (A.9), so is that vector multiplied by any scalar. However, because the **r**'s are linearly independent, the *m*'s and thus also the combined factors *m***r** are uniquely determined by (A.15)

cases in subsequent sections.

A.3 Complex Eigenvalues

In general, the eigenvalues, eigenvectors and behavior modes may not be real numbers. However, because the coefficients in the matrix **G** are all real, the complex eigenvalues and their associated eigenvectors always appear in conjugate pairs. We may therefore remove the complex numbers as follows. Consider two complex conjugate eigenvalues

$$\begin{aligned}\lambda_1 &= \alpha + i\omega,\\ \lambda_2 &= \alpha - i\omega.\end{aligned} \qquad \text{(A.18)}$$

Then the corresponding terms in (A.16) are

$$\mathbf{w_1} e^{(\alpha+i\omega)t} + \mathbf{w_2} e^{(\alpha-i\omega)t} = (\mathbf{c} - i\,\mathbf{h}) e^{(\alpha+i\omega)t} + (\mathbf{c} + i\,\mathbf{h}) e^{(\alpha-i\omega)t}, \qquad \text{(A.19)}$$

where the **c** and **h** are real-valued vectors. Using the identity $e^{iz} = cosz + isinz$ and collecting terms, we get the expression

$$e^{\alpha t}\left[(2\mathbf{c} + i(-\mathbf{h} + \mathbf{h}))cos\omega t + (2\mathbf{h} + i(\mathbf{c} - \mathbf{c}))sin\omega t\right], \qquad \text{(A.20)}$$

which amounts to the following real vector

$$e^{\alpha t}\left[2\mathbf{c}\ cos\omega t + 2\mathbf{h}\ sin\omega t\ \right] \qquad \text{(A.21)}$$

i.e. the expression reduces to

$$e^{\alpha t}\ 2(\mathbf{c}\ cos\omega t + \mathbf{h}\ sin\omega t). \qquad \text{(A.22)}$$

In our analysis, we have chosen an equivalent, in our opinion more intuitive, form of (A.22),

$$e^{\alpha t}\mathbf{a}\ sin(\omega t + \boldsymbol{\theta}), \qquad \text{(A.23)}$$

where the vector components $a_i, \theta_i, i = 1, \ldots, n$ are determined by

$$\begin{aligned}a_i &= 2\sqrt{c_i^{\,2} + h_i^{\,2}},\\ \theta_i &= arctan\frac{c_i}{h_i} + \begin{cases}0 & if\ h_i \geq 0\\ \pi & if\ h_i < 0\end{cases}\end{aligned} \qquad \text{(A.24)}$$

A.4 Zero eigenvalues

If one of the eigenvalues of **G**, say λ_1, is zero, the expression for the corresponding weight in (A.17) is no longer valid. It is easy to see, however, that a zero eigenvalue contributes with a constant term, $e^{0t}\mathbf{r}_1 = \mathbf{r}_1$, in (A.14). When integrated with respect to time, this constant term contributes with a linear growth mode yielding the solution

$$\mathbf{x}(t) = \mathbf{w_0} + \mathbf{w_1}t + \mathbf{w_2}e^{\lambda_1 t} + \ldots + \mathbf{w_n}e^{\lambda_n t} \tag{A.25}$$

where

$$\begin{aligned} \mathbf{w_0} &= -\mathbf{w_2} - \ldots - \mathbf{w_n}, \\ \mathbf{w_1} &= m_1(0)\mathbf{r_1}, \\ \mathbf{w_k} &= \frac{m_k(0)\mathbf{r_k}}{\lambda_k}, \quad k = 2,\ldots,n. \end{aligned} \tag{A.26}$$

A.5 Non-distinct eigenvalues

We now relax the assumption that all the eigenvalues are all distinct. The eigenvalues are found as the roots of the polynomial equation $|\mathbf{G} - \lambda\mathbf{I}| = 0$. An eigenvalue that is a p-fold root to this equation is said to have *multiplicity p.* It is now no longer certain that this eigenvalue will have p linearly independent associated eigenvectors, which is required for the transformation (A.10). It is then necessary to employ *generalized eigenvectors.* A generalized eigenvector **v** of rank k of the matrix **G** associated with an eigenvalue λ is defined by the property

$$\begin{aligned} (\mathbf{G} - \lambda\mathbf{I})^k \mathbf{v} &= 0, \\ (\mathbf{G} - \lambda\mathbf{I})^{k-1} \mathbf{v} &\neq 0. \end{aligned} \tag{A.27}$$

Note that for k=1, (A.27) reduces to

$$\begin{aligned} (\mathbf{G} - \lambda\mathbf{I})\mathbf{v} &= 0, \\ \mathbf{v} &\neq 0, \end{aligned} \tag{A.28}$$

which is the definition of an ordinary eigenvector.

Now assume, without loss of generality, that the matrix **G** has a single eigenvalue λ, which thus has multiplicity n.[4] Assume also that the matrix $\mathbf{G}-\lambda\mathbf{I}$ has rank n-1, i.e., there is only one linearly independent associated eigenvector, $\mathbf{r_1}$.[5] A set of n generalized eigenvectors may be found in recursive manner by successively solving

$$\begin{aligned}(\mathbf{G}-\lambda\mathbf{I})\mathbf{r_1} &= 0,\\ (\mathbf{G}-\lambda\mathbf{I})\mathbf{r_2} &= \mathbf{r_1},\\ (\mathbf{G}-\lambda\mathbf{I})\mathbf{r_3} &= \mathbf{r_2},\\ &\vdots\\ (\mathbf{G}-\lambda\mathbf{I})\mathbf{r_n} &= \mathbf{r_{n-1}}.\end{aligned} \tag{A.29}$$

It can be shown both that such a solution exists and that the generalized eigenvectors $\mathbf{r_1},\ldots,\mathbf{r_n}$ are linearly independent (Chen 1970, p. 40). We may now use this set of vectors as a basis in the model state space and repeat the analysis in section A.2, where differentiating (A.10) now yields

$$\begin{aligned}\ddot{\mathbf{x}}(t) &= \dot{m}_1(t)\mathbf{r_1}+\cdots+\dot{m}_n(t)\mathbf{r_n}\\ &= \mathbf{G}\left(m_1(t)\mathbf{r_1}+\cdots+m_n(t)\mathbf{r_n}\right)\\ &= m_1(t)\mathbf{G}\mathbf{r_1}+\cdots+m_n(t)\mathbf{G}\mathbf{r_n}\\ &= m_1(t)\lambda\mathbf{r_1}+m_2(t)(\lambda\mathbf{r_2}+\mathbf{r_1})+\cdots+m_n(t)(\lambda\mathbf{r_n}+\mathbf{r_{n-1}})\\ &= (\lambda m_1(t)+m_2(t))\mathbf{r_1}+\cdots+(\lambda m_{n-1}(t)+m_n(t))\mathbf{r_{n-1}}+\lambda m_n(t)\mathbf{r_n},\end{aligned} \tag{A.30}$$

leading to the differential equations for the modes m,

$$\begin{aligned}\dot{m}_1(t) &= \lambda m_1(t)+m_2(t),\\ &\vdots\\ \dot{m}_{n-1}(t) &= \lambda m_{n-1}(t)+m_n(t),\\ \dot{m}_n(t) &= \lambda m_n(t),\end{aligned} \tag{A.31}$$

which can be solved recursively to yield

[4] If **G** has other eigenvalues, these simply contribute extra terms in the solution. For notational convenience, we have chosen to ignore these terms, since they do not affect the solution for the repeated eigenvalue.

[5] If more than one linearly independent vector exists for the same eigenvalue, the procedure in the following is repeated for each vector. However, this condition implies that two or more of the modes are exactly equal and the system therefore exhibits less than n different behavior modes. We believe, but have not yet verified, that the only way this can happen is if parts of the system are uncoupled from each other, something that is rare in system dynamics models because of the endogenous viewpoint underlying the methodology.

$$
\begin{aligned}
m_n(t) &= m_n(0)e^{\lambda t}, \\
m_{n-1}(t) &= \left(m_{n-1}(0) + m_n(0)t\right)e^{\lambda t}, \\
m_{n-2}(t) &= \left(m_{n-2}(0) + m_{n-1}(0)t + m_n(0)\tfrac{1}{2}t^2\right)e^{\lambda t}, \\
&\vdots \\
m_k(t) &= e^{\lambda t}\sum_{i=0}^{n-k} m_{k+i}(0)\frac{t^i}{i!}, \\
&\vdots \\
m_1(t) &= e^{\lambda t}\sum_{i=0}^{n-1} m_{i+1}(0)\frac{t^i}{i!}.
\end{aligned}
\qquad \text{(A.32))}
$$

As before, setting $t = 0$, yields

$$
\begin{aligned}
\dot{\mathbf{x}}(0) &= m_1(0)\mathbf{r_1} + \cdots + m_n(0)\mathbf{r_n} \\
&= \mathbf{b},
\end{aligned}
\qquad \text{(A.33)}
$$

where the fact that the generalized eigenvectors are linearly independent assures a unique solution for the $m_k(0)$. The solution for the time derivative is then

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= m_1(t)\mathbf{r_1} + \cdots + m_n(t)\mathbf{r_n} \\
&= e^{\lambda t}\sum_{i=0}^{n-1} m_{i+1}(0)\frac{t^i}{i!}\mathbf{r_1} + \cdots + e^{\lambda t}\sum_{i=0}^{n-k} m_{k+i}(0)\frac{t^i}{i!}\mathbf{r_k} + \cdots \\
&\quad + e^{\lambda t}\left(m_{n-1}(0) + m_n(0)t\right)\mathbf{r_{n-1}} + e^{\lambda t}m_n(0)_n^0\mathbf{r_n} \\
&= e^{\lambda t}\begin{bmatrix} \left(m_1(0)\mathbf{r_1} + \cdots + m_k(0)\mathbf{r_k} + \cdots m_{n-1}(0)\mathbf{r_{n-1}} + m_n(0)\mathbf{r_n}\right) + \\ \left(m_2(0)\mathbf{r_1} + \cdots + m_{k+1}(0)\mathbf{r_k} + \cdots + m_n(0)\mathbf{r_{n-1}}\right)\frac{t}{1!} + \cdots + m_n(0)\mathbf{r_1}\frac{t^{n-1}}{(n-1)!} \end{bmatrix} \\
&= e^{\lambda t}\left[\mathbf{s_1} + \mathbf{s_2}t + \cdots \mathbf{s_n}t^{n-1}\right],
\end{aligned}
\qquad \text{(A.34)}
$$

where

$$
\mathbf{s_k} = \sum_{i=1}^{n-k+1} m_{i+k-1}(0)\mathbf{r_i},\ k = 1,\ldots,n. \qquad \text{(A.35)}
$$

Integrating (A.34) yields the solution

$$
\mathbf{x}(t) = \mathbf{w_0} + \mathbf{w_1}e^{\lambda t} + \mathbf{w_2}te^{\lambda t} + \ldots + \mathbf{w_j}t^{j-1}e^{\lambda t} + \ldots + \mathbf{w_n}t^{n-1}e^{\lambda t}, \qquad \text{(A.36)}
$$

where

$$
\begin{aligned}
\mathbf{w_k} &= \frac{1}{(k-1)!}\sum_{i=0}^{n-k}\frac{\mathbf{s_{k+i}}}{\lambda^{i+1}}(-1)^i\,(k+i-1)!, \quad k = 1,\ldots,n, \\
\mathbf{w_0} &= -\mathbf{w_1}.
\end{aligned}
\qquad \text{(A.37)}
$$