Topological Quantum Computation

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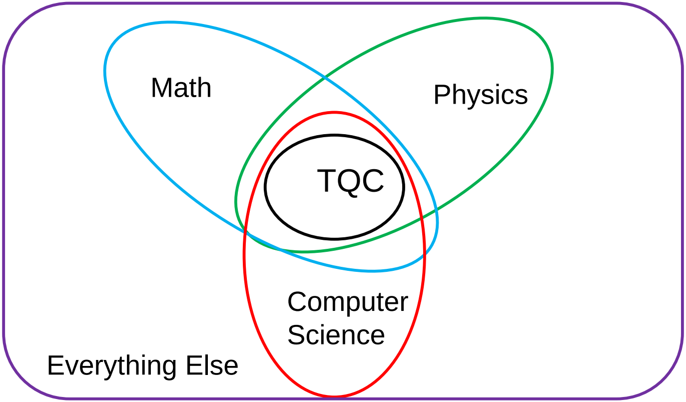
Topological Quantum Computation



What is a Quantum Computer?

From [Freedman-Kitaev-Larsen-Wang '03]:

Definition

Quantum Computation is any computational model based upon the theoretical ability to manufacture, manipulate and measure quantum states.

Prototypical Quantum Computation Scheme

Fix a (classical) function $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$.

1. Goal: compute $f(N)$.
2. Encode classical information N as a quantum state $|N\rangle$.
3. Process state: $|N\rangle \to \sum_j a_j |j\rangle$ depending on f.
4. Measure state: get $|j\rangle$ with probability $|a_j|^2$, hopefully $|a_{f(N)}|^2 \gg 0$.

A Universal Quantum Circuit Model

Let $V = \mathbb{C}^2$.

Example

- **state space** (n-qubit): $V^{\otimes n}$
- **quantum gate set**: $U_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,

  $U_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix}$, $U_3 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- **quantum circuits**: $\prod_j I_V^{\otimes a_j} \otimes U_{i_j} \otimes I_V^{\otimes b_j} \in U(V^{\otimes n})$, $1 \leq i_j \leq 3$.

Theorem

Universal: n-qubit quantum circuits **dense** *in* $SU(V^{\otimes n})$.

Remarks on QCM

Remarks

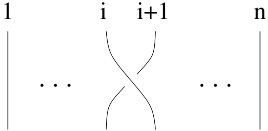
- Typical physical realization: composite of n identical d-level systems. E.g. $d = 2$: spin-$\frac{1}{2}$ arrays.
- The setting of most quantum algorithms: e.g. Shor's integer factorization algorithm
- Main nemesis: decoherence–errors due to interaction with surrounding material. Requires expensive error-correction...

Question

How to overcome decoherence? One answer: **TOPOLOGY**.

The Braid Group

A key role is played by the braid group $\mathcal{B}_n$ generated by σ_i:



Definition (Artin)

$\mathcal{B}_n$ is generated by σ_i, $i = 1, \ldots, n-1$ satisfying:

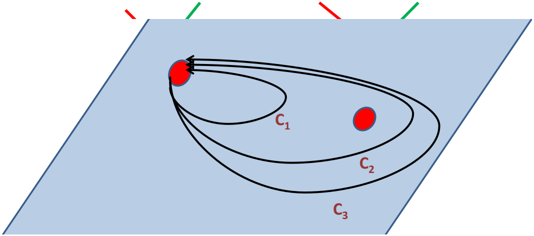
(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$

Anyons

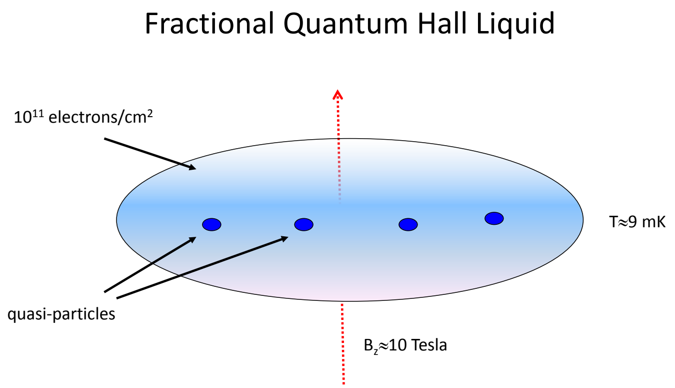
For Point-like particles:

- In $\mathbb{R}^3$: bosons or fermions: $\psi(z_1, z_2) = \pm\psi(z_2, z_1)$
- Particle exchange $\rightsquigarrow$ reps. of symmetric group S_n
- In $\mathbb{R}^2$: anyons: $\psi(z_1, z_2) = e^{i\theta}\psi(z_2, z_1)$
- Particle exchange $\rightsquigarrow$ reps. of braid group $\mathcal{B}_n$
- Why? $\pi_1(\mathbb{R}^3 \setminus \{z_i\}) = 1$ but $\pi_1(\mathbb{R}^2 \setminus \{z_i\}) = F_n$ Free group.



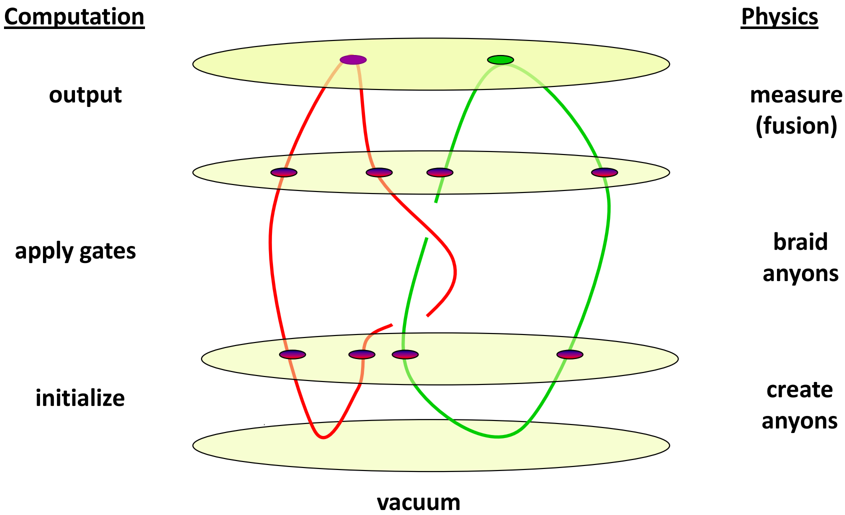
$C_1 \not\approx C_2 \approx C_3$

Topological Phases of Matter/Anyons



Topological Quantum Computation (TQC) is a computational model built upon systems of topological phases.

Topological Model



Mathematical Problems

1. Model for anyonic systems/topological phases
2. Classify (models of) topological phases
3. Interpret information-theoretic questions
4. 3-dimensional generalizations?

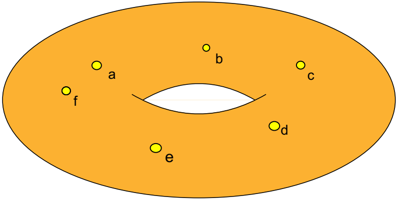
Modeling Anyons on Surfaces

Topology of marked surfaces+quantum mechanics“Marks” (boundary components) are labelled by anyons, of which there are finitely many (distinguishable, indecomposable).

Principle

Superposition: a state is a vector in a Hilbert space $|\psi\rangle \in \mathcal{H}$.

Interpretation

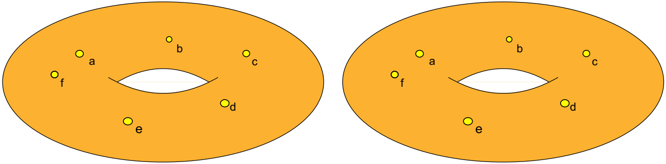


$\rightarrow \mathcal{H}$

Principle

The **composite state space** of two physically separate systems A and B is the **tensor product** $\mathcal{H}_A \otimes \mathcal{H}_B$ of their state spaces.

Interpretation



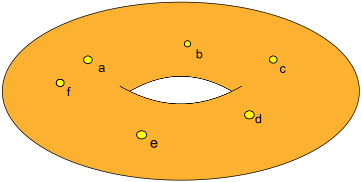
$\rightarrow \mathcal{H} \otimes \mathcal{H}$

Principle

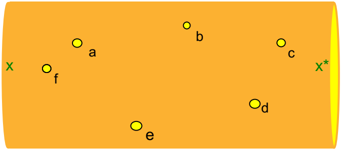
Locality: the global state is determined from local information (on disks, plus boundary conditions).

Interpretation

The Hilbert space of a marked surface M is a direct sum over all boundary labelings of a surface M_g obtained by cutting M along a circle.



$\mathcal{H} =$



$\bigoplus_x \mathcal{H}_x$
(x^* is anti-particle to x)
.

Definition (Nayak, et al '08)

a system is in a topological phase if its low-energy effective field theory is a topological quantum field theory (TQFT).

A 3D **TQFT** assigns to any surface (+boundary data ℓ) a Hilbert space:

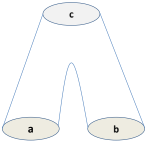
$$(M, \ell) \to \mathcal{H}(M, \ell).$$

Each boundary circle $\bigcirc$ is labelled by $i \in \mathcal{L}$ a finite set of "colors". $0 \in \mathcal{L}$ is **neutral**. Orientation-reversing map: $x \to x^*$.

Basic pieces

Any surface can be built from the following basic pieces:

- disk: $\mathcal{H}($  $; i) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & else \end{cases}$
- annulus: $\mathcal{H}($  $; a, b) = \begin{cases} \mathbb{C} & a = b^* \\ 0 & else \end{cases}$
- pants:

$P :=$ 

$$\mathcal{H}(P; a, b, c) = \mathbb{C}^{N(a,b,c)}$$

Two more axioms

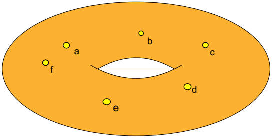
Axiom (Disjoint Union)

$\mathcal{H}[(M_1,\ell_1)\coprod(M_2,\ell_2)] = \mathcal{H}(M_1,\ell_1)\otimes\mathcal{H}(M_2,\ell_2)$

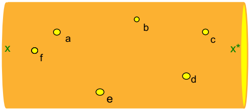
Axiom (Gluing)

If M is obtained from gluing two boundary circles of M_g together then

$$\mathcal{H}(M,\ell) = \bigoplus_{x\in\mathcal{L}} \mathcal{H}(M_g,\ell,x,x^*)$$



(M,ℓ)



(M_g,ℓ,x,x^*)

Algebraic Part=Modular Category

Morally, (2+1)TQFTs=Modular Categories:

Definition

A **modular category** $\mathcal{C}$ (over $\mathbb{C}$) is

monoidal: $(\otimes, \mathbf{1})$,

semisimple: $X \cong \bigoplus_i m_i X_i$,

linear: $\mathrm{Hom}(X, Y) \in Vec_{\mathbb{C}}$,

rigid: $X^* \otimes X \mapsto \mathbf{1} \mapsto X \otimes X^*$,

finite rank: $Irr(\mathcal{C}) = \{\mathbf{1} = X_0, \ldots, X_{r-1}\}$,

spherical: $u_X \theta_X : X \cong X^{**}$, $\dim(X) \in \mathbb{R}$,

braided: $c_{X,Y} : X \otimes Y \cong Y \otimes X$,

modular: $Det(Tr_{\mathcal{C}}(c_{X_i,X_j^*} c_{X_j^*,X_i})) = Det(S_{ij}) \neq 0$.

Key Data

- fusion rules: $X_i \otimes X_j \cong \bigoplus_k N_{ij}^k X_k$
- fusion ring representation: $X_i \to N_i$ where $(N_i)_{k,j} = N_{i,j}^k$
- (modular) S-matrix: $S_{ij} := \mathrm{Tr}_{\mathcal{C}}(c_{X_i,X_j^*} c_{X_j^*,X_i})$
- (Dehn twist) T-matrix: $T_{ij} = \delta_{ij}\theta_i$
- (quantum)Dimensions: $\dim(X_i) := S_{i0}$ if unitary, $\dim(X_i) = \max \mathrm{Spec}(N_i)$.

(2+1)TQFT Anyon Model vs Modular Category

Each axiom has a corresponding physical interpretation:

TQFT/anyonic system	Category $\mathcal{C}$
anyon types $x \in \mathcal{L}$	simple X
vacuum $0 \in \mathcal{L}$	$\mathbf{1}$
x^* antiparticle	dual X^*
$\mathcal{H}(P; x, y, z)$ state space	$\mathrm{Hom}(X \otimes Y, Z)$
particle exchange	braiding $c_{X,X}$
anyon types observable	$\det(S) \neq 0$
topological spin/Dehn twist	θ_X

Example: Fibonacci Theory

- $\mathcal{L} = \{0, 1\}$
- pants: $\mathcal{H}(P; a, b, c) = \begin{cases} \mathbb{C} & a = b = c \\ \mathbb{C} & a + b + c \in 2\mathbb{Z} \\ 0 & else \end{cases}$
- Define: $V_k^i := \mathcal{H}(D^2 \setminus \{z_i\}_{i=1}^k; i, 1, \cdots, 1)$
- $\dim V_n^i = \begin{cases} Fib(n-2) & i = 0 \\ Fib(n-1) & i = 1 \end{cases}$

Classification

Question (Physics)

How many models exist for a given fixed number of distinguishable indecomposable anyon types?

Theorem (Bruillard,Ng,R,Wang JAMS '15)

There are finitely many modular categories of any given rank r.

Proof.

Recall Richard Ng's colloquium at USC on March 12, 2014. □

$2 \leq$rank≤ 5 fusion rules (Hong,Ng,Bruillard,Wang,Stong,R.):

$\lvert\mathcal{L}\rvert$	$\mathcal{C}$
2	$PSU(2)_3$, $SU(2)_1$
3	$\mathbb{Z}_3$, $PSU(2)_7$, $SU(2)_2$
4	products, $\mathbb{Z}_4$, $PSU(2)_9$
5	$\mathbb{Z}_5, PSU(2)_{11}, SU(3)_4/\mathbb{Z}_3, SU(2)_4$

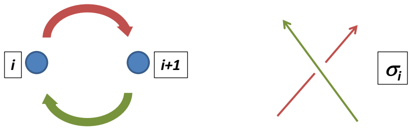
Braid group representations

$\mathcal{B}_n$ acts on **state spaces**:

- Fix anyons x, y
- Braid group acts linearly:

$$\mathcal{B}_n \curvearrowright \mathcal{H}(D^2 \setminus \{z_i\}; x, \cdots, x, y) = \mathrm{Hom}(X^{\otimes n}, Y)$$

by particle exchange



Universal Anyons

Question (Quantum Information)

When does an anyon x provide universal computation models?
Basically: when is $\mathcal{B}_n \curvearrowright \mathrm{Hom}(X^{\otimes n}, Y)$ dense?

Example

Fibonacci $\dim(X) = \frac{1+\sqrt{5}}{2}$ is universal: braid group $\mathcal{B}_n$ image is dense in $SU(F_n) \times SU(F_{n-1})$

Example

Ising $\dim(X) = \sqrt{2}$ is not universal: braid group $\mathcal{B}_n$ image is a finite group.

Characterization of Universal anyons

Conjecture (R '07,property F)

Anyon x is universal if, and only if, $\dim(X)^2 \notin \mathbb{Z}$.

Theorem

The property **F** *conjecture is:*

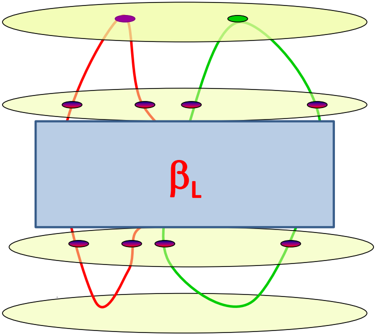
- *(R,Wenzl '14) true for quantum groups*
- *(Etingof,R,Witherspoon '08) true for group-theoretical categories.*

What do TQCs naturally compute?

Answer

(Approximations to) Link invariants!

Associated to $x \in \mathcal{L}$ is a link invariant $Inv_L(x)$ approximated by the corresponding Topological Model efficiently.



Prob( ) ~ $x^{-t}|Inv_L($ $)|$

Complexity of Jones Polynomial Evaluations

$V_L(q)$ Jones polynomial at $q = e^{2\pi i/\ell}$

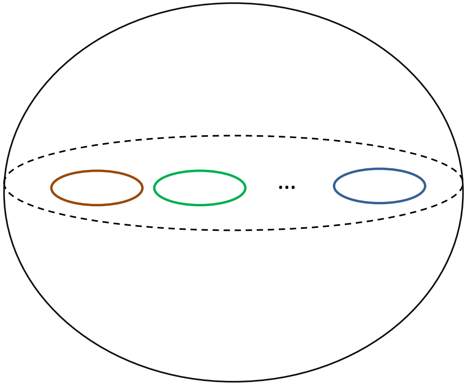
Theorem (Vertigan,Freedman-Larsen-Wang)

- *(Classical) exact computation of $V_L(q)$ at $q = e^{2\pi i/\ell}$ is:*
  $$\begin{cases} FP & \ell = 3,4,6 \\ FP^{\sharp P} - complete & else \end{cases}$$
- *(Quantum) approximation of $|V_L(q)|$ at $q = e^{2\pi i/\ell}$ is BQP.*

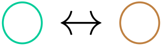
3-dimensional materials

- Point-like particles in $\mathbb{R}^3$
- ~~Point-like particles~~ in $\mathbb{R}^3$

loop-like particles?

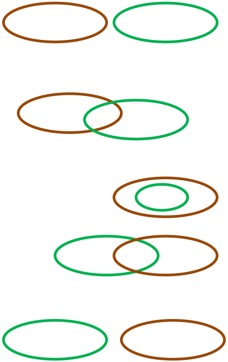


Two operations:

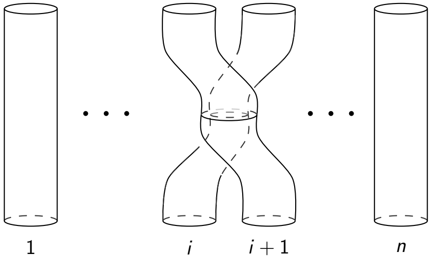
Loop interchange s_i: 

and Leapfrogging (read upwards):

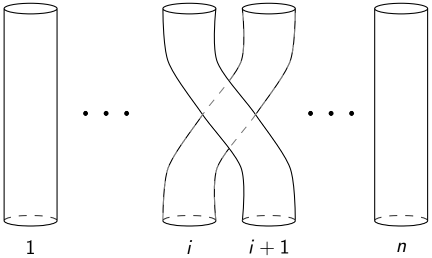
σ_i :



$\sigma_i =$



$s_i =$



The Loop Braid Group $\mathcal{LB}_n$ is generated by $s_1, \ldots, s_{n-1}, \sigma_1, \ldots, \sigma_{n-1}$ satisfying:

Braid relations:

(R1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

(R2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$

Symmetric Group relations:

(S1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

(S2) $s_i s_j = s_j s_i$ if $|i - j| > 1$

(S3) $s_i^2 = 1$

Mixed relations:

(M1) $\sigma_i \sigma_{i+1} s_i = s_{i+1} \sigma_i \sigma_{i+1}$

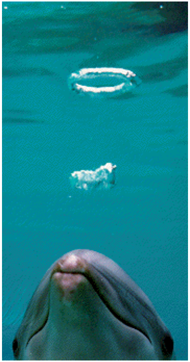
(M2) $s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1}$

(M3) $\sigma_i s_j = s_j \sigma_i$ if $|i - j| > 1$

Question

Mathematical models? Do these exist in nature?

$(3+1)$TQFTs...



References

- Chang,R.,Plavnik,Sun,Bruillard,Hong: 1508.00005 (J. Math. Phys.)
- Kadar,Martin,R.,Wang: 1411.3768 (Glasgow J. Math.)
- Bruillard,Ng,R.,Wang: 1310.7050 (J. Amer. Math. Soc.)
- Naidu,R.: 0903.4157 (J. Alg. Rep. Theory)
- Z. Wang "Topological Quantum Computation" Amer. Math. Soc. 2008.

Thank you!