Work on Inverse Sturm-Liouville Problems

The classical inverse Sturm-Liouville is to recover the potential q in

$$-u'' + q(x)u = \lambda u, \qquad u'(0) - hu(0) = 0, \quad u'(L) + Hu(L) = 0$$

from spectral data; consisting of the eigenvalues $\{\lambda_n\}$ and some additional information on the associated eigenfunctions $\{u_n(x)\}$. The first comprehensive result was due to Borg [1], who showed that if $\{\lambda_n\}$, $\{\mu_n\}$ are the eigenvalue sequences corresponding to two values H, H' of the impedance parameter with $H \neq H'$ and where h is fixed, then $q(x)$ is uniquely determined (as is both H, H'). An alternative is to take the normalisation $u'(0) = 1$ if $u(0) - 0$ and $u(1) = 0$ if $u'(0) = 0$ then prescribe a single spectrum $\{\lambda_n\}$ together with the *norming constants* $\{\rho_n = \|u_n\|_2^2\}$. This is the formulation of Gel'fand-Levitan [2]. For a survey/short history see the monograph [3].

[1] G. Borg, Eine Umkehrung der Sturm-Liouville Eigenwertaufgabe, *Acta Mathematica,* 1946, **76**, 1–96.

[2] I. M. Gel'fand and B. M. Levitan, On the determination of a differential equation from its spectral function, *Amer. Math. Soc. Transl.*, 1951, **1**, 253–291.

[3] K. Chadan, D. Colton, L. Päivärinta, W. Rundell, An Introduction to Inverse Scattering and Inverse Spectral Problems, *SIAM Monographs on Mathematical Modeling and Computation*, 1997,

There are many other combinations of spectral information and the shortest proofs and probably the fastest reconstruction method for q uses the Gel'fand Levitan transformation and the spectral data to convert the problem to a nonlinear Volterra integral equation as shown in [1]. There are of course generalisations of the original equation, boundary conditions and to more complex geometries.

The references listed below are my publications in the area.

[1] William Rundell and Paul E. Sacks. Reconstruction techniques for classical inverse Sturm-Liouville problems. *Math. Comp.*, 58(197):161–183, 1992.

[2] William Rundell. Recovering the density of a string from only lowest frequency data. *SIAM J. Appl. Math.*, 75(5):2232–2245, 2015.

[3] William Rundell and Paul E. Sacks. Inverse eigenvalue problem for a simple star graph. *J. Spectr. Theory*, 5(2):363–380, 2015.

[4] William Rundell and Paul E. Sacks. An inverse eigenvalue problem for a vibrating string with two Dirichlet spectra. *SIAM J. Appl. Math.*, 73(2):1020–1037, 2013.

[5] Bangti Jin and William Rundell. An inverse Sturm-Liouville problem with a fractional derivative. *J. Comput. Phys.*, 231(14):4954–4966, 2012.

[6] William Rundell and Paul E. Sacks. Numerical technique for the inverse resonance problem. *J. Comput. Appl. Math.*, 170(2):337–347, 2004.

[7] C. Maeve McCarthy and William Rundell. Eigenparameter dependent inverse Sturm-Liouville problems. *Numer. Funct. Anal. Optim.*, 24(1-2):85–105, 2003.

[8] William Rundell and Paul E. Sacks. Reconstruction of a radially symmetric potential from two spectral sequences. *J. Math. Anal. Appl.*, 264(2):354–381, 2001.

[9] Bruce D. Lowe and William Rundell. An inverse problem for a Sturm-Liouville operator. *J. Math. Anal. Appl.*, 181(1):188–199, 1994.

[10] William Rundell and Paul E. Sacks. The reconstruction of Sturm-Liouville operators. *Inverse Problems*, 8(3):457–482, 1992.

[11] Bruce D. Lowe, Michael Pilant, and William Rundell. The recovery of potentials from finite spectral data. *SIAM J. Math. Anal.*, 23(2):482–504, 1992.

[12] Joyce R. McLaughlin and William Rundell. A uniqueness theorem for an inverse Sturm-Liouville problem. *J. Math. Phys.*, 28(7):1471–1472, 1987.