

My research interests have always revolved around various types of inverse problems for partial differential equations. There are three basic mathematical issues:

- is there a unique solution?
- how ill-conditioned is the inverse operator and how can this be resolved?
- what computational methods are available and do these give convergence to a solution?

In the last few years I have been interested in reaction diffusion equations and systems including the case of the underlying pde incorporating a non-local operator such as arises with fractional derivatives. The reasons for this are two-fold. First, these operators frequently provide a better physical model than one based on classical assumptions on derivatives and integrals. Second, inverse problems are almost always ill-conditioned and finding solution methods means resolving the inversion of an unbounded operator; replacing the classical equation by a nonlocal one can be an effective method – the idea of *quasi-reversibility*. Most of the current work is joint with Barbara Kaltenbacher.

Examples in the first category are numerous and a prominent case is to use the subdiffusion operator $\partial_t^\alpha u - \mathbb{L}u$ where $-\mathbb{L}$ is a uniformly elliptic operator in $\Omega \subset \mathbb{R}^n$ in place of the standard parabolic model. The forwards regularity of this operator is much weaker and the analysis more complex. One main effect is replacement of the exponential time decay of the parabolic by a linear decay for the subdiffusion case, and this often has positive consequences for inverse problems. See the survey article [1] for an overview of inverse problems involving such derivatives. Recent papers highlighting the recovery of inclusions and unknown coefficients and the dependence on α are [2, 3, 4, 5]. The question of recovering α obviously arises and in fact there might be a finite sum of such with orders $\{\alpha_i\}_1^N$. For the general fully distributed derivative case and its recovery from temporal data see [6].

Another direction is the damped wave equation. The undamped case is $u_{tt} - c^2 \mathbb{L}u = 0$ and the long term behaviour is purely sinusoidal in nature. For the classically damped equation $u_{tt} - bu_t - c^2 \mathbb{L}u = 0$ all solutions decay exponentially and thus all frequencies decay at this rate. This latter effect does not correspond to the physics in many cases and thus the model $u_{tt} - b \partial_t^\alpha u - c^2 \mathbb{L}u = 0$ where $\partial_t^\alpha u$ is a fractional derivative of order α , $0 < \alpha < 1$ is often used. In this case, for all frequencies, the decay is only a polynomial of small degree in time. The paper [7] gives an application of this idea to recovering, for example, the wave speed $c(x)$ under both classical and fractional operators.

An example of the second type is to recover the initial condition from a final time measurement for a parabolic equation. This is perhaps the most well-known severely ill-posed inverse problem and many regularisation methods have been proposed. One such, involves replacing the equation by a subdiffusion operator of order α and letting $\alpha \rightarrow 1$ This fails for the important very low frequencies and a more sophisticated version can be found in [8].

Reaction-diffusion equations are the backbone of chemical reactions, ecology models and just about any physical model that balances a diffusion process with a (often nonlinear) driving term. The canonical form is $u_t - \Delta u = f(u)$ where Δ is the Laplacian and a more general setting would extend this to an elliptic operator $\mathbb{L} = -\nabla(a(x)\nabla u) + q(x)u$. There are many formulations:

- The reaction $f(u)$ is known and we must recover the coefficients, $a(x)$ and $q(x)$; [9, 10]
- The operator \mathbb{L} is known and we have to recover $f(u)$; [11]
- both $a(x)$ and $f(u)$ are unknown, [12], $a(u)$ is unknown, [13].

Overposed data to accomplish this can be time-trace boundary measurements or values at a time T .

There is also the case of reaction-diffusion systems

$$u_t - \mathbb{L}u = f_1(u) + g_1(\phi(u, v)) \quad v_t - \mathbb{L}v = f_2(v) + g_2(\phi(u, v)).$$

The interaction terms g_1, g_2 and the coupling $\phi(u, v)$ might be known and both f_1 and f_2 have to be recovered. Alternatively, if the f_i are known then we seek to recover both g_1 and g_2 or if each f_i and g_i are known, recover the coupling term ϕ . See [14] for both analysis and reconstructions. For the case of coefficient recovery in a damped wave equation see [7, 15] and in a fully nonlinear problem (the Westervelt equation), see [16, 17, 18].

Recent publications in the area

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- [3] William Rundell and Zhidong Zhang. Recovering an unknown source in a fractional diffusion problem. *J. Comput. Phys.*, 368:299–314, 2018.
- [4] William Rundell and Zhidong Zhang. On the identification of source term in the heat equation from sparse data. *SIAM J. Math. Anal.*, 52(2):1526–1548, 2020.
- [5] William Rundell and Masahiro Yamamoto. Uniqueness for an inverse coefficient problem for a one-dimensional time-fractional diffusion equation. *Applicable Analysis*, (10.1080/00036811.2021.1965583), 2021.
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- [8] Barbara Kaltenbacher and William Rundell. Regularization of a backwards parabolic equation by fractional operators. *Inverse Problems and Imaging*, 13:401–430, 2019.
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- [10] Barbara Kaltenbacher and William Rundell. Recovery of multiple coefficients in a reaction-diffusion equation. *Journal of Mathematical Analysis and Applications*, 481:123475, 2019.
- [11] Barbara Kaltenbacher and William Rundell. On the identification of a nonlinear term in a reaction-diffusion equation. *Inverse Problems*, 35:115007, 2019.
- [12] Barbara Kaltenbacher and William Rundell. On the simultaneous recovery of the conductivity and the nonlinear reaction term in a parabolic equation. *Inverse Probl. Imaging*, 14(5):939–966, 2020.
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- [15] Barbara Kaltenbacher and William Rundell. Determining damping terms in fractional wave equations. *Inverse Problems*, 38, 2022.
- [16] Barbara Kaltenbacher and William Rundell. On the identification of the nonlinearity parameter in the westervelt equation from boundary measurements. *Inverse Probl. Imaging*, 2021.
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