Discrete Structures for Computing

CSCE 222

Sandeep Kumar

Many slides based on [Lee19], [Rog21], [GK22]

Counting

Chapter 6

Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations
- Generating Permutations and Combinations

The Basics of Counting

Section 6.1

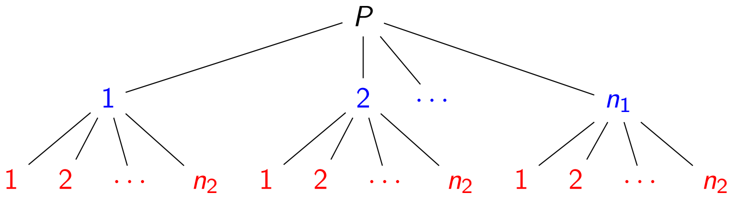
- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Tree Diagrams

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \times n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.



The Product Rule

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

By the product rule, there are

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

different possible license plates.

$$\underbrace{\_\_\ \ \_\_\ \ \_\_}_{\text{26 choices each}} \quad \underbrace{\_\_\ \ \_\_\ \ \_\_}_{\text{10 choices each}}$$

Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are

$$n \times n \times \cdots \times n = n^m$$

such functions.

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements?

Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and $n - 1$ ways to choose a_2, etc. The product rule tells us that there are

$$n(n-1)(n-2)\cdots(n-m+1)$$

such functions.

Telephone Numbering Plan

The North American numbering plan (NANP) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let X denote a digit from $0 \ldots 9$.
- Let N denote a digit from $2 \ldots 9$.
- Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was NYX-NNX-XXXX.
- In the new plan, the format is NXX-NXX-XXXX.

Telephone Numbering Plan...

How many different telephone numbers are possible under the old plan and the new plan?

Use the Product Rule.

- There are $8 \times 2 \times 10 = 160$ area codes with the format NYX
- There are $8 \times 8 \times 10 = 640$ office codes with the format NNX.
- There are $10 \times 10 \times 10 \times 10 = 10,000$ station codes with the format $XXXX$.

Number of old plan telephone numbers:

$$160 \times 640 \times 10,000 = 1,024,000,000$$

Number of new plan telephone numbers:

$$800 \times 800 \times 10,000 = 6,400,000,000$$

Counting Subsets of a Finite Set

Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$. In Section 5.1, we used mathematical induction to prove this same result.

When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. When the i^{th} element is in the subset, the bit string has a 1 in the i^{th} position and a 0 otherwise.

By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Consider the set $S = \{a, b, c\}$. Then

$$\phi \equiv 000$$
$$\{a\} \equiv 100, \{b\} \equiv 010, \{c\} \equiv 001$$
$$\{a, b\} \equiv 110, \{a, c\} \equiv 101, \{b, c\} \equiv 011$$
$$\{a, b, c\} \equiv 111$$

Product Rule in Terms of Sets

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

The task of choosing an element in the Cartesian product

$$A_1 \times A_2 \times \cdots \times A_m$$

is done by choosing an element in A_1, an element in A_2,..., and an element in A_m.

By the product rule, it follows that:

$$|A1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$$

DNA and Genomes

- A *gene* is a segment of a DNA molecule that encodes a particular protein and the entirety of genetic information of an organism is called its genome.
- DNA molecules consist of two strands of blocks known as nucleotides. Each nucleotide is composed of bases: adenine (A), cytosine (C), guanine (G), or thymine (T).
- The DNA of bacteria has between 10^5 and 10^7 links (one of the four bases). Mammals have between 10^8 and 10^{10} links. So, by the product rule there are at least 4^{10^5} different sequences of bases in the DNA of bacteria and 4^{10^8} different sequences of bases in the DNA of mammals.
- The human genome includes approximately 23,000 genes, each with 1,000 or more links.
- Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

Basic Counting Principles: The Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are

$$n_1 + n_2$$

ways to do the task.

Example: The mathematics department must choose

- either a student, or
- a faculty member as a representative for a university committee.

How many choices are there for this representative if there are

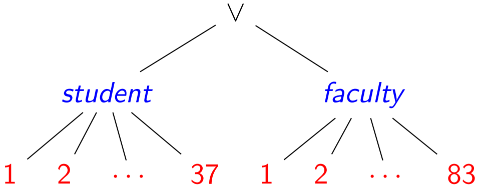
- 37 members of the mathematics faculty, and
- 83 mathematics majors and
- no one is both a faculty member and a student.

By the sum rule it follows that there are

$$37 + 83 = 120$$

possible ways to pick a representative.

The Sum Rule as a picture



The Sum Rule in terms of sets

The sum rule can be phrased in terms of sets.

$$|A \cup B| = |A| + |B|$$

as long as A and B are disjoint sets. Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

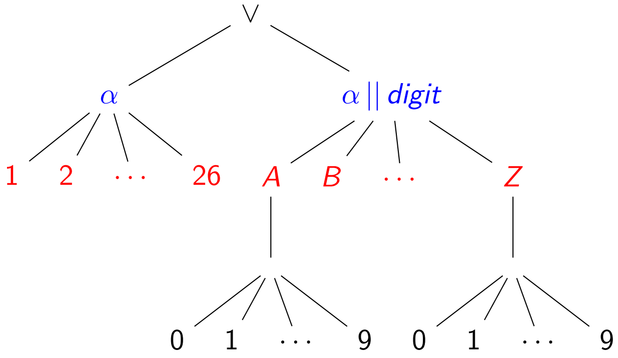
when $A_i \cap A_j = \varnothing, \ \forall i, j$.

The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Combining the Sum and Product Rule

Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Use the product rule: $26 + 26 \times 10 = 286$.



Counting Passwords

Combining the sum and product rule allows us to solve more complex problems.

Each user on a computer system has a password,

- Which is six to eight characters long,
- Each character is an uppercase letter or a digit.
- Each password must contain at least one digit.
- How many possible passwords are there?

Counting Passwords...

Let P be the total number of passwords, and let $P_6, P_7, \& P_8$ be the passwords of length $6, 7, \& 8$.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, P_8, we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:
  - $P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$
  - $P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920.$
  - $P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.$
- Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360.$

Internet Addresses

Version 4 of the Internet Protocol (IPv4) uses 32 bits.

- **Class A Addresses**: used for the largest networks, a 0, followed by a 7-bit netid and a 24-bit hostid.
- **Class B Addresses**: used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid.
- **Class C Addresses**: used for the smallest networks, a 110, followed by a 21-bit netid and an 8-bit hostid.

- Neither Class D nor Class E addresses are assigned as the address of a computer on the Internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Host-ids consisting of all 0s and all 1s are not available in any network.

Counting Internet Addresses

<table>
<tr><td>Bit #</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>0</td><td>1</td></tr>
<tr><td>Class A</td><td>0</td><td colspan="7">netid</td><td colspan="24">hostid</td></tr>
<tr><td>Class B</td><td>1</td><td>0</td><td colspan="14">netid</td><td colspan="16">hostid</td></tr>
<tr><td>Class C</td><td>1</td><td>1</td><td>0</td><td colspan="21">netid</td><td colspan="8">hostid</td></tr>
<tr><td>Class D</td><td>1</td><td>1</td><td>1</td><td>0</td><td colspan="28">multicast address</td></tr>
<tr><td>Class E</td><td>1</td><td>1</td><td>1</td><td>1</td><td>0</td><td colspan="27">reserved for future use</td></tr>
</table>

Counting Internet Addresses

How many different IPv4 addresses are available for computers on the internet?

Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

- To find, $x_A : 2^7 - 1 = 127$ netids. $2^{24} - 2 = 16,777,214$ hostids. $x_A = 127 \times 16,777,214 = 2,130,706,178$.
- To find, $x_B : 2^{14} = 16,384$ netids. $2^{16} - 2 = 16,534$ hostids. $x_B = 16,384 \times 16,534 = 1,073,709,056$.
- To find, $x_C : 2^{21} = 2,097,152$ netids. $2^8 - 2 = 254$ hostids. $x_C = 2,097,152 \times 254 = 532,676,608$.
- Hence, the total number of available IPv4 addresses is

$$\begin{aligned} x &= x_A + x_B + x_C \\ &= 2,130,706,178 + 1,073,709,056 + 532,676,608 \\ &= 3,737,091,842 \end{aligned}$$

Not enough addresses today! The newer IPv6 protocol solves the problem of too few addresses.

Basic Counting Principles: Subtraction Rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Also known as, the principle of inclusion-exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

How many bit strings of length *eight* either start with a 1 bit or end with the two bits 00?

- Use the subtraction rule. Number of bit strings of length eight that start with a 1 bit:

$$\underline{1}\ \_\ \_\ \_\ \_\ \_\ \_\ \_ = 2^7 = 128$$

- Number of bit strings of length eight that end with bits 00:

$$\_\ \_\ \_\ \_\ \_\ \_\ \underline{0}\ \underline{0} = 2^6 = 64$$

- Number of bit strings of length 8 that start with a 1 bit and end with bits 00:

$$\underline{1}\ \_\ \_\ \_\ \_\ \_\ \underline{0}\ \underline{0} = 2^5 = 32$$

- Hence, the number is $128 + 64 - 32 = 160$.

Basic Counting Principles: Division Rule

There are n/d *ways* to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then

$$n = |A|/d$$

In terms of functions: If f is a function from $A \to B$, where both are finite sets, and for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$, then $|B| = |A|/d$.

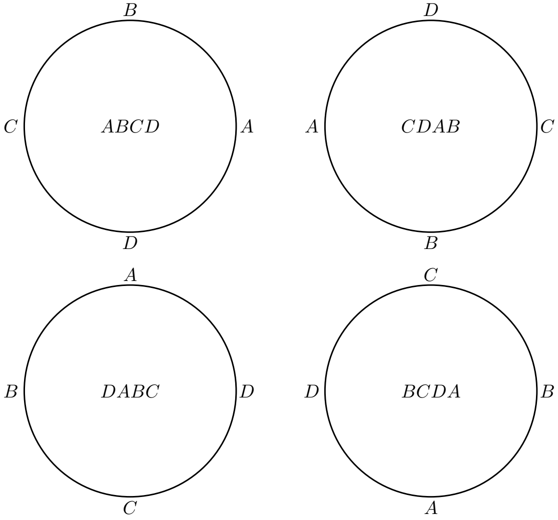
Basic Counting Principles: Division Rule...

How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

- Number the seats around the table from 1 to 4 proceeding clockwise.
- There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4.
- Thus there are $4! = 24$ ways to order the four people.
- But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are $24/4 = 6$ different seating arrangements.

Basic Counting Principles: Division Rule. . .

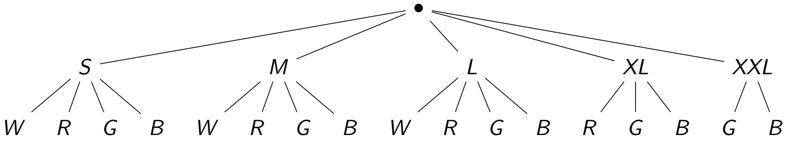


Basic Counting Principles: Division Rule. . .

Tree Diagrams

Tree Diagrams: We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.

Example: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S, M, L, XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?



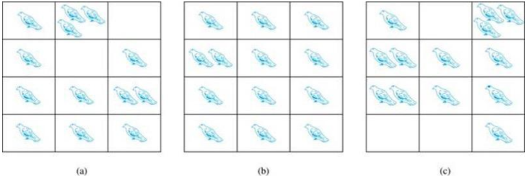
The store must stock 17 T-shirts.

Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



Pigeonhole Principle: If k is a positive integer and $k+1$ objects are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition.

- ($k+1$ objects in k boxes) $\rightarrow$ (at least one box has ≥ 2 objects).
- (No box has ≥ 2 objects) $\rightarrow$ ($\leq k$ objects in k boxes).
  - Suppose none of the k boxes has more than one object.
  - Then the total number of objects would be at most k.
  - This contradicts the statement that we have $k+1$ objects.

The Pigeonhole Principle

Corollary 1: A function f from a set with $k+1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f.
- Put in the box for y all of the elements x from the domain such that $f(x) = y$.
- Because there are $k+1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one.

Pigeonhole Principle

- Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
- Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion. E.g., $2 \times 5 = 10, 3 \times 37 = 111$.
  - Let n be a positive integer.
  - Consider the $n+1$ integers $1, 11, 111, \ldots$, where the last # has $(n+1)$ ones.
  - There are n possible remainders when an integer is divided by n.
  - By the pigeonhole principle, when each of the $n+1$ integers is divided by n, at least two must have the same remainder.
  - Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion.
- Consider the number 3.
  - Consider the numbers: $1, 11, 111, 1111$.
  - Remainders mod 3: $1, 2, 0, 1$. So $(3 \mid 1111 - 1)$ which comprises of only 0s and 1s.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k\left(\lceil N/k \rceil - 1\right) < k\left((N/k + 1) - 1\right) = N$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used. This is a contradiction because there are a total of N objects.

Example: Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

The Generalized Pigeonhole Principle

- How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
  - We assume four boxes; one for each suit.
  - Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards.
  - At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$.
  - The smallest integer N such that

$$\lceil N/4 \rceil \geq 3 = 2 \times 4 + 1 = 9$$

- How many must be selected to guarantee that at least three hearts are selected?
  - A deck contains 13 hearts and 39 cards which are not hearts.
  - So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts.
  - However, when we select 42 cards, we must have at least three hearts.
  - Note that the generalized pigeonhole principle is not used here.

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permutation*.

Example: Let $S = \{1, 2, 3\}$.

- The ordered arrangement $3, 1, 2$ is a permutation of S.
- The ordered arrangement $3, 2$ is a 2-permutation of S.

The number of r-permutations of a set with n elements is denoted by $P(n, r)$.

- The 2-permutations of $S = \{1, 2, 3\}$ are:
  - $1, 2$; $1, 3$; $2, 1$; $2, 3$; $3, 1$; and $3, 2$.
  - Hence, $P(3, 2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n-1$ ways, and so on until there are $(n-(r-1))$ ways to choose the last element.

Note that $P(n, 0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$ then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

$$P(100, 3) = 100 \cdot 99 \cdot 98 = 970,200$$

Solving Counting Problems by Counting Permutations

Suppose that a saleswoman

- Has to visit eight different cities.
- She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes.
- How many possible orders can the saleswoman use when visiting these cities?

The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations

- How many permutations of the letters *ABCDEFGH* contain the string *ABC*?
- We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a binomial coefficient.

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

$C(4, 2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of r-combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n, r) = \frac{n!}{(n-r)!\, r!}$$

Proof: By the product rule $P(n, r) = C(n, r) \cdot P(r, r)$. Therefore,

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}$$

Combinations

- How many poker hands of five cards can be dealt from a standard deck of 52 cards?
- Also, how many ways are there to select 47 cards from a deck of 52 cards?

Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!\,5!} = C(52,5) = 2,598,960$$

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then

$$C(n, r) = C(n, n - r)$$

Proof: From Theorem 2, it follows that

$$\begin{aligned} C(n, r) &= \frac{n!}{r!\,(n-r)!} \\ C(n, n-r) &= \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!} \end{aligned}$$

Hence, $C(n, r) = C(n, n - r)$.

Combinatorial Proofs

Definition 1: A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *bijective* proof shows that there is a bijection between the sets of objects counted by the two sides of the identity.
- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.

Combinatorial Proofs

Here are two combinatorial proofs that

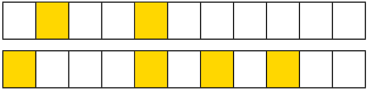
$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

- *Bijective Proof*: Suppose that S is a set with n elements.
  - Let $f : A \in S \to \bar{A}$.
  - f is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements.
  - Implies that the two sets have the same number of elements.
- *Double Counting Proof*: By definition, the number of subsets of S with r elements is $C(n, r)$.
  - Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$.
  - Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements.

Combinatorial Proofs

- For every set of gold cells, there's a set of white cells—a bijection.



- Instead, just count the number of sets with gold cells.
  - They are ≡ the number of sets with white cells—counting.

Combinations

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!\,5!} = 252$$

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!\,24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775$$

Section 6.4 Summary

- The Binomial Theorem
- Pascal's Identity and Triangle
- Other Identities Involving Binomial Coefficients

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. More generally, these terms can be products of constants and variables.

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.

Powers of Binomial Expressions

- To obtain xy^2, an x must be chosen from one of the sums and a y from the other two. There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
- To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

We have used a counting argument to show that

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Next we present the binomial theorem which gives the coefficients of the terms in the expansion of $(x+y)^n$.

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then $(x+y)^n =$

$$\sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Proof: We use combinatorial reasoning.

- The terms in the expansion of $(x+y)^n$ are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$.
- To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums.
- Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.

Section 6.5 Summary

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition.

. . . .

- How many strings of length r can be formed from the uppercase letters of the English alphabet?
- The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

How many ways are there to select five bills from a box containing at least five of each of the following denominations:

$$\$1, \$2, \$5, \$10, \$20, \$50, \text{ and } \$100$$

Place the selected bills in the appropriate position of a cash box illustrated below:

100	50	20	10	5	2	1

Combinations with Repetition

Some possible ways of placing the five bills:

		2				3
100	50	20	10	5	2	1

$\equiv \,|\,|\bullet\bullet|\,|\,|\,|\bullet\bullet\bullet$

1	1	2		1		
100	50	20	10	5	2	1

$\equiv \bullet|\bullet|\bullet\bullet|\,|\bullet|\,|$

- The number of ways to select five bills corresponds to the number of ways to arrange six 1s and five 0s in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!\,6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n+r-1, r) = C(n+r-1, n-1)$$

Proof Sketch: We need distinct permutations of $n+r-1$ items where $n-1$ items are of type A, and the remaining items are of type B.

Combinations with Repetition

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, x_3 are nonnegative integers?

. . . .

x_1	x_2	x_3

Combinations with Repetition

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

. . . .

The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

Type	Repetition Allowed?	Formula
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

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