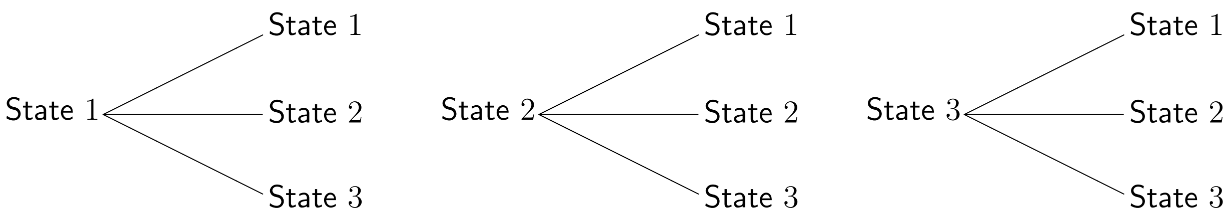
9.1: Markov Chains

DEFINITION 1. ***Markov process***, *or* ***Markov Chain****, is an experiment consisting of a finite number of stages in which the outcomes and associated probabilities of each stage depend only on the outcome of the preceding stage. The outcomes are called* **states** *and the outcome of the current experiment is referred to as the* **current state** *of the process.*

The probability of going from one state to another state on the next trial depends *only* on the present experiment and not on past history.



$$\begin{aligned} a_{ij} &= \mathrm{P}(\text{State } i | \text{State } j) \\ &= \text{transition probability from State } j \text{ (\textit{current} state) to State } i \text{ (\textit{next} state)} \end{aligned}$$

DEFINITION 2. *A* ***transition matrix*** *associated with a Markov Chain with n states is an $n \times n$ matrix T with entries a_{ij} having the following properties:*

1. $a_{ij} \geq 0$ *for all i and j.*
2. *The sum of the entries in each column of T is 1.*

Any square matrix satisfying properties (1) and (2) is referred to as a ***stochastic matrix****.*

$$T = \text{Next state} \begin{array}{c} \\ \text{State 1} \\ \text{State 2} \\ \text{State 3} \end{array} \overset{\text{Current state}}{\begin{array}{ccc} \text{State 1} & \text{State 2} & \text{State 3} \\ \end{array}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

EXAMPLE 3. *Determine which of the matrices are stochastic:*

(a) $\begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.3 \\ 0.4 & 0.6 & 0.3 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0.2 & 0.4 \\ 0 & -0.2 & 0.3 \\ 0 & 0.8 & 0.3 \end{bmatrix}$

EXAMPLE 4. *At a certain university, three bookstores - the University Bookstore (U), the Campus Bookstore (C), and the Book Mart (M) - currently serve the university community. Each quarter the the University Bookstore retains* 80% *of its customers but loses* 10% *to the Campus Bookstore and* 10% *to the Book Mart. The Campus Bookstore retains* 75% *of its customers but loses* 10% *to the University Bookstore and* 15% *to the Book Mart. The Book Mart retains* 90% *of its customers but loses* 5% *to the University Bookstore and* 5% *to the Campus Bookstore.*

Give a tree diagram and the transition matrix.

From a survey conducted at the beginning of the fall quarter, it was found that the University Bookstore and the Campus Bookstore each had 40% *of the market, whereas the Book Mart had* 20% *of the market. Assuming the buying habits of the customers of these bookstores stay the same for the future (i.e. these trends continue) answer the following questions:*

(a) *What percentage of the market will the University Bookstore have at the beginning of the second quarter?*

(b) *What percentage of the market will the Campus Bookstore have at the beginning of the second quarter?*

(c) *What percentage of the market will the Book Mart have at the beginning of the second quarter?*

(d) *What percentage of the market will each store have at the beginning of the third quarter? The fourth quarter?*

Now let us do the previous Example with matrices:

The transition matrix is $T = \begin{array}{c} \\ U \\ C \\ M \end{array} \begin{array}{c} \begin{array}{ccc} U & C & M \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.05 \\ 0.1 & 0.75 & 0.05 \\ 0.1 & 0.15 & 0.9 \end{bmatrix} \end{array}$.

The **initial state matrix** is $X_0 = \begin{array}{c} U \\ C \\ M \end{array} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$.

We get the distribution at the beginning of the second quarter (i.e. after 1 quarter), X_1 by multiplying the transition matrix, T, by the initial-state matrix, X_0:

$$X_1 = TX_0 = \begin{bmatrix} 0.8 & 0.1 & 0.05 \\ 0.1 & 0.75 & 0.05 \\ 0.1 & 0.15 & 0.9 \end{bmatrix} \cdot \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.37 \\ 0.35 \\ 0.28 \end{bmatrix}$$

The distribution after 1 quarter is given by: $X_1 = \begin{array}{c} U \\ C \\ M \end{array} \begin{bmatrix} 0.37 \\ 0.35 \\ 0.28 \end{bmatrix}$.

So, the University Bookstore has 37% of the market, the Campus Bookstore has 35% of the market, and the Book Mart has 28% of the market. Same answers we got when solving it with the tree diagrams.

Now, *What percentage of the market will each store have at the beginning of the third quarter?*

What percentage of the market will each store have at the beginning of the fourth quarter?

REMARK 5. In general, if the initial distribution is represented by matrix X_0, the distribution after m observations can be found as follows:

$$\begin{array}{rclc} X_1 & = & TX_0 & \\ X_2 & = & TX_1 & = T(TX_0) = \\ X_3 & = & TX_2 & = T(T^2X_0) = \\ \cdot & \cdot & \cdot & \\ X_m & = & T^mX_0 & \end{array}$$

EXAMPLE 6. *The transition matrix for a Markov process is given by*

$$T = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix}$$

(a) *What does the entry* $a_{11} = 0.4$ *represent?*

(b) *Given that the outcome state 1 has occurred, what is the probability that the next outcome of the experiment will be state 2?*

(c) *If the initial-state distribution vector is given by* $X_0 = \begin{bmatrix} 0.15 \\ 0.85 \end{bmatrix}$ *the probability distribution of the system after one observation.*

EXAMPLE 7. *Find the probability distribution of the system after three observations for the distribution vector* $X_0 = \begin{bmatrix} 0.15 \\ 0.4 \\ 0.45 \end{bmatrix}$ *and the transition matrix* $T = \begin{bmatrix} 0.1 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.5 \\ 0.8 & 0.7 & 0.2 \end{bmatrix}$.

EXAMPLE 8. *At the beginning of 1990, the population of a certain state was* 70% *rural and* 30% *urban. Based on past trends, it is expected that* 10% *of the population currently residing in the rural areas will move into the urban areas, while* 40% *of the population currently residing in the urban areas will move into the rural areas in the next ten years.*

(a) *What is the transition matrix?*

(b) *What was the population distribution in the state at the beginning of 2000?*

(c) *What was the population distribution in the state at the beginning of 2010?*