Section 2.3: Calculating limits using the limits laws

LIMIT LAWS Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
3. $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a} c = c$
6. $\lim_{x\to a} x = a$
7. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, where n is a positive integer.
8. $\lim_{x\to a} x^n = a^n$, where n is a positive integer.
9. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer and if n is even, then we assume that $\lim_{x\to a} f(x) > 0$.
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{\lim_{x\to a} x}$ where n is a positive integer and if n is even, then we assume that $a > 0$.

REMARK 1. Note that *all these properties also hold for the one-sided limits.*

REMARK 2. The analogues of the *laws 1-3 also hold when f and g are vector functions* (the product in Law 3 should be interpreted as a dot product).

EXAMPLE 3. *Compute the limit:*
$\lim_{x\to -2} \frac{x^2+x+1}{x^3-10} =$

REMARK 4. The function from Example 3 also satisfies "direct substitution property":

$$\lim_{x\to a} f(x) = f(a).$$

Later we will say that such functions are *continuous.* Note that in both examples it was important that a in the domain of f.

EXAMPLE 5. *Compute the limit:*

$$\lim_{x\to 1} \frac{x-1}{x^2-4x+3}$$

EXAMPLE 6. *Given*

$$g(x) = \begin{cases} x^2+4, & \text{if} \quad x \le -1 \\ 2-3x & \text{if} \quad x > -1 \end{cases}$$

Compute the limits:

(a) $\lim_{x\to 4} g(x)$

(b) $\lim_{x\to -1} g(x)$

EXAMPLE 7. *Evaluate these limits.*

(a) $\lim_{x\to 4} \frac{x^{-1}-0.25}{x-4}$

(b) $\lim_{x\to 0} \frac{(x+5)^2-25}{x}$

(c) $\lim_{x\to 0^-} \left\{ \frac{1}{x} - \frac{1}{|x|} \right\}$

(d) $\lim_{x\to -1} \frac{|x+1|}{x+1}$

(e) $\lim_{x\to 0} \frac{\sqrt{6-x}-\sqrt{6}}{x}$

Conclusion from the above examples:

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To calculate the limit of f(x) as x → a:
PLUG IN x = a if a is in the domain of f.
Otherwise "FACTOR" or "MULTIPLY BY CONJUGATE" and then plug in.
Consider one sided limits if necessary.
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Squeeze Theorem. Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$g(x) \le f(x) \le h(x)$$

and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$. Then

$$\lim_{x\to a} f(x) = L.$$

Corollary. Suppose that for all x in an interval containing a (except possibly at $x = a$)

$$|f(x)| \le h(x) \quad (\textit{equivalently}, \quad -h(x) \le f(x) \le h(x))$$

and $\lim_{x\to a} h(x) = 0$. Then

$$\lim_{x\to a} f(x) = 0.$$

EXAMPLE 8. *Given* $3x \le f(x) \le x^3 + 2$ *for* $0 \le x \le 2$. *Find* $\lim_{x\to 1} f(x)$

EXAMPLE 9. *Evaluate:*

(a) $\lim_{x\to 0} x \sin \frac{1}{x}$

(b) $\lim_{t\to 0} (t^5) \cos^3 (\frac{1}{t^2})$

DEFINITION 10. *A function f is called* ***bounded*** *on an open interval I, if there exists a number M such that $|f(x)| \le M$ for all x in I.*

EXAMPLE 11. *Let f be a bounded function on an open interval I containing the point $x = c$ and g be a function defined on I, but not necessarily at $x = c$. Find* $\lim_{x\to c} (f(x)g(x))$ *if it is given that* $\lim_{x\to c} g(x) = 0$.