Section 2.4: The Precise Definition of a Limit

Question: What does it mean $\lim_{x\to a} f(x) = L$? To motivate the precise definition of limit, consider the function

$$f(x) = \begin{cases} 2x-1, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}$$

- What is $\lim_{x\to 3} f(x)$?
- **Problem 1** *How close to* 3 *does* x *have to be so that* $f(x)$ *differs from* 5 *by less than* 0.1*?*
  - The distance from x to 3 is ____________
  - The distance from $f(x)$ to 5 is ____________
- Reformulation of problem 1: *Find a number* δ *such that*

Thus an answer to the Problem 1 is given by $\delta =$ ______; that is, if x is within a distance of ______ from 3, then $f(x)$ will be within a distance of ______ from 5.

- **Problem 2** *How close to* 3 *does* x *have to be so that* $f(x)$ *differs from* 5 *by less than* 0.01*?*

- **Problem 3** *How close to* 3 *does* x *have to be so that* $f(x)$ *differs from* 5 *by less than* 0.001*?*

- **Problem 4** *How close to* 3 *does* x *have to be so that* $f(x)$ *differs from* 5 *by less than an arbitrary positive number* ε*?*

$$|f(x)-5| < \varepsilon \quad \text{if} \quad 0 < |x-3| < \delta = \frac{\varepsilon}{2}. \tag{1}$$

In other words, we can make the values of $f(x)$ within an arbitrary distance ε from 5 by taking the values of x within a distance $\varepsilon/2$ from 3 (but $x \neq 3$). This is a precise way of saying that $f(x)$ is close to 5 when x is close to 3. Note that (1) can be rewritten as follows:

DEFINITION 1. *Let $f(x)$ be a function defined for all x in some open interval containing the number a, except possibly at a itself. Then we say that* **the limit of $f(x)$ as x approaches a is L**, *and we write*

$$\lim_{x\to a} f(x) = L,$$

if for every number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad 0 < |x - a| < \delta.$$

REMARK 2. For a limit from the right we need only assume that $f(x)$ is defined on an interval (a, b) extending to the right of a and that the ϵ condition is met for x in an interval $a < x < a + \delta$ extending to the right of a. A similar adjustment must be made for a limit from the left.

A general form of a limit proof

Assume that we are given a positive number ϵ, and we try to prove that we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

There are two things to do:

1. Preliminary analysis of the problem (guessing a value for δ);
2. Proof (showing that the δ works).

Note that *the value of δ is not unique.* Namely, once we have found a value of δ that fulfills the requirements of the definition, then any *smaller* positive number $\delta_1, \delta_1 < \delta$, will also fulfill those requirements.

EXAMPLE 3. *Use the "epsilon-delta" definition to prove that* $\lim_{x\to 4}(3x-1)=11$.

EXAMPLE 4. *Use the "epsilon-delta" definition to prove that* $\lim_{x\to 5} x^2=25$.