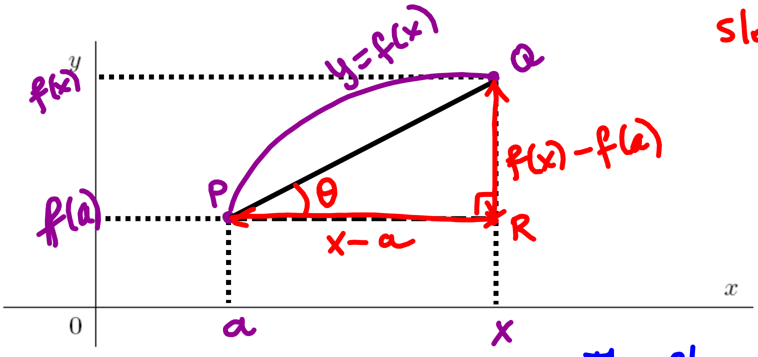
Section 2.7:Tangents, velocities, and other rates of change.

Consider a curve with equation $y = f(x)$ and points $P(a, f(a))$ and $Q(x, f(x))$ on it. The slope of the secant line PQ (also known as **average rate** or **average velocity**) is

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

$$\text{slope} = \tan\theta = \frac{\overline{QR}}{\overline{PR}}$$



Note

$$m_{PQ} = \frac{f(a) - f(x)}{a - x}$$

The slope of tangent at P:

$$\lim_{Q \to P} m_{PQ}$$

DEFINITION 1. *The* **tangent line** *to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

provided that this limit exists.

The slope (1) of tangent line also known as **instantaneous rate of change** or **instantaneous velocity**.

EXAMPLE 2. (a) *Find the slope of the tangent line to the graph of* $f(x) = x^2 - 3x - 4$ *at* $(5, 6)$.

$a = 5$

$f(a) = f(5) = 6$

$$m = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5} \frac{x^2 - 3x - 4 - 6}{x - 5}$$

$$= \lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x \to 5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = 5 + 2 = \boxed{7}$$

(b) *Find the equation of the tangent line to the graph of* $f(x)$ *at* $x = 5$.(Recall that point-slope form for a line through the point (x_1, y_1) with slope m is: $\boxed{y - y_1 = m(x - x_1)}$)

$m = 7$

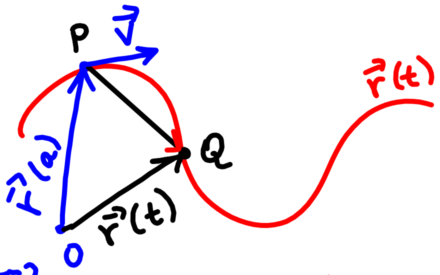
$x_1 = 5$

$y_1 = f(x_1) = 6$

$$\boxed{y - 6 = 7(x - 5)}$$

DEFINITION 3. *If* $\mathbf{r}(t)$ *is a vector function then the* **tangent vector v** *at* $t=a$ *is found by*

$$\mathbf{v}=\lim_{t\to a}\underbrace{\frac{1}{t-a}}_{\text{scalar}}\underbrace{[\mathbf{r}(t)-\mathbf{r}(a)]}_{\text{vector function}}.$$



By Triangle Law

$\vec{OP}+\vec{PQ}=\vec{OQ}$

$\vec{r}(a)+\vec{PQ}=\vec{r}(t)$

$\vec{PQ}=\vec{r}(t)-\vec{r}(a)$

$Q\to P$ as $t\to a$

EXAMPLE 4. *Given curve* $\mathbf{r}(t)=\langle 2t, 10t-t^2\rangle$.

instanteneous velocity at t=2

(a) *Find a vector tangent to the curve at the point* $(4,16)$.

$a=2$ (Because $\vec{r}(2)=\langle 4,16\rangle$)

$$\vec{v}=\lim_{t\to 2}\frac{1}{t-2}\left[\vec{r}(t)-\vec{r}(2)\right]=$$

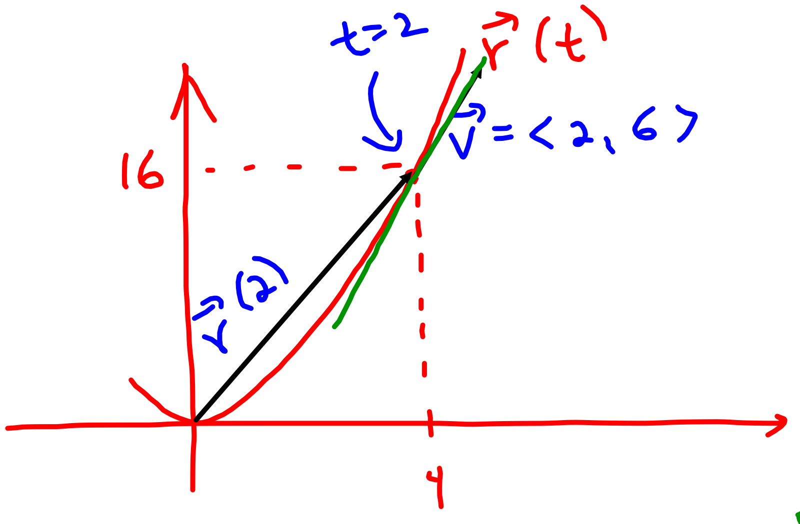
$$=\lim_{t\to 2}\frac{1}{t-2}\left[\langle 2t, 10t-t^2\rangle-\langle 4,16\rangle\right]$$

$$=\lim_{t\to 2}\frac{1}{t-2}\langle 2t-4, 10t-t^2-16\rangle$$

$$=\lim_{t\to 2}\left\langle\frac{2\cancel{(t-2)}}{\cancel{t-2}}, -\frac{\cancel{(t-2)}(t-8)}{\cancel{t-2}}\right\rangle$$

$$=\lim_{t\to 2}\langle 2, -(t-8)\rangle=\lim_{t\to 2}\langle 2, 8-t\rangle=\langle 2, 8-2\rangle=\boxed{\langle 2,6\rangle}$$

$10t-t^2-16$
$=-(t^2-10t+16)$
$=-(t-2)(t-8)$



$$x = x_0 + at$$
$$y = y_0 + bt$$

coordinates of $\vec{r}(t_0)$

components of $\vec{v}$

(b) *Find parametric equations of the tangent line to* $\mathbf{r}(t)$ *at* $t = 2$.

$$\boxed{\begin{aligned} x &= 4 + 2t \\ y &= 16 + 6t \end{aligned}}$$

(c) *Find a cartesian equation of this tangent line.*

Way I

$$t = \boxed{\frac{x-4}{2} = \frac{y-16}{6}}$$

Way 2

slope $m = \frac{6}{2} = 3$

$$\boxed{y - 16 = 3(x - 4)}$$

Velocities. Denote by $f(t)$ the position of an object at time t.

The **Average Velocity** of the object from $t = a$ to $t = b$ is

$$v_{ave} = \frac{f(b) - f(a)}{b - a}.$$

The **Instantaneous Velocity** of the object at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(Compare this formula with $\underline{(1)}$ by substitution $\underbrace{h = x - a}$.)

$$\boxed{h = x - a \Rightarrow x = a + h \xrightarrow[h \to 0]{} a}$$

$$\downarrow$$

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{h}$$

EXAMPLE 5. *The position (in meters) of an object moving in a straight path is given by*

$$s(t) = t^2 - 2t + 8,$$

where t is measured in seconds.

$S(5) = 25 - 10 + 8 = 23$

$\boxed{S(4) = 16 - 8 + 8 = 16}$

(a) *Find the average velocity over the time interval* $[4, 5]$.

$$v_{ave} = \frac{S(5) - S(4)}{5 - 4} = \frac{23 - 16}{1} = \boxed{7 \ m/s}$$

(b) *Find the instantaneous velocity at time* $\overset{=a}{t} = 4$.

$$v(4) = \lim_{h \to 0} \frac{S(a+h) - S(a)}{h} = \lim_{h \to 0} \frac{S(4+h) - S(4)}{h}$$

$$= \lim_{h \to 0} \frac{(4+h)^2 - 2(4+h) + 8 - 16}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{16} + 8h + h^2 - \cancel{8} - 2h + \cancel{8} - \cancel{16}}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} (6 + h) = \boxed{6 \ m/s}$$

Other Rates of Change:

The **Average Rate** of change of function $f(x)$ from $x = a$ to $x = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The **Instantaneous Rate of Change** of $f(x)$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$