Section 3.10: Related rates

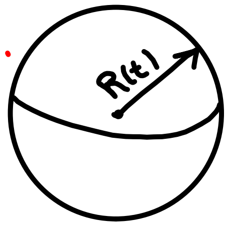
0. Identify

In this section, we have two or more quantities that are changing with respect to time t. We will apply the following strategy:

1. Read the problem carefully and draw a diagram if possible.
2. Express the given information and the required rates in terms of derivatives and state your "find" and "when".
3. Find a formula (equation) that relates the quantities in the problem. (If necessary, use Geometry[1] of the situation to eliminate one of the variables by substitution.) **Don't substitute the given numerical information at this step!!!**
4. Use the Chain Rule to differentiate both sides of the equation with respect to t.
5. Substitute the given numerical information in the resulting equation and solve for the desired rate of change.

EXAMPLE 1. *A spherical balloon is inflated with gas at a rate of* $25\text{ft}^3/\text{min}$. *How fast is the radius changing when the radius is* 2ft*?*

0. Quantities: volume $V(t)$
   radius $R(t)$

1. 

2. Find $R'(t)$ when $R(t) = 2\text{ ft}$
   $V'(t) = 25\text{ ft}^3/\text{min}$

3. We know that for sphere: $V = \frac{4\pi}{3} R^3$

   In our case $V(t) = \frac{4\pi}{3}\left(R(t)\right)^3$

4. Differentiate both sides using Chain Rule:

$$V'(t) = \frac{4\pi}{\cancel{3}}\,\cancel{3}\left(R(t)\right)^2 \cdot R'(t)$$

5. Substitute the given numerical data:

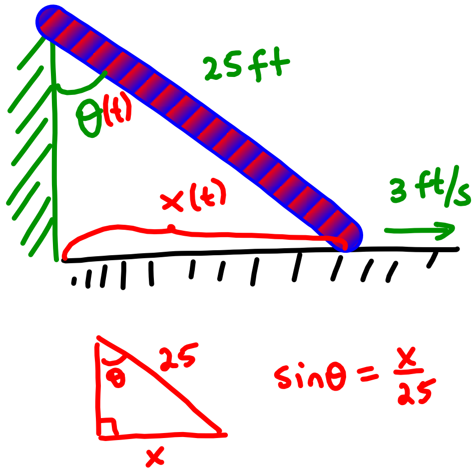
$$25 = 4\pi \cdot 2^2\, R'(t)$$

$$R'(t) = \boxed{\frac{25}{16\pi}\text{ ft/min}}$$

[1] Useful formulas:

- Triangle: $A = \frac{1}{2}bh$
  - Equilateral Triangle: $h = \dfrac{\sqrt{3}s}{2}$; $A = \dfrac{\sqrt{3}s^2}{4}$
  - Right Triangle: Pythagorean Theorem $c^2 = a^2 + b^2$
- Trapezoid: $A = \dfrac{h}{2}(b_1 + b_2)$
- Parallelogram: $A = bh$
- Circle: $A = \pi r^2$; $C = 2\pi r$
- Sector of Circle: $A = \frac{1}{2}r^2\theta$; $s = r\theta$
- Sphere: $V = \dfrac{4}{3}\pi r^3$; $A = 4\pi r^2$
- Cylinder: $V = \pi r^2 h$
- Cone: $V = \dfrac{1}{3}\pi r^2 h$

EXAMPLE 2. *A ladder 25 feet long and leaning against a vertical wall. The bottom of the ladder slides away from the wall at speed* 3 feet/sec. *Determine how fast the angle between the top of the ladder and the wall is changing when the angle is* $\frac{\pi}{4}$ *radians.*



Quantities: $\theta(t)$, $x(t)$

Given $\frac{dx}{dt} = 3$ ft/sec

Find $\frac{d\theta}{dt}$ when $\theta = \pi/4$

$$\sin\theta(t) = \frac{x(t)}{25}$$

Differentiate and use C-R:

$$\frac{d}{dt}\left(\sin\theta(t)\right) = \frac{d}{dt}\left(\frac{x(t)}{25}\right)$$

$$\cos\theta(t)\,\frac{d\theta}{dt} = \frac{1}{25}\,\frac{dx}{dt}$$

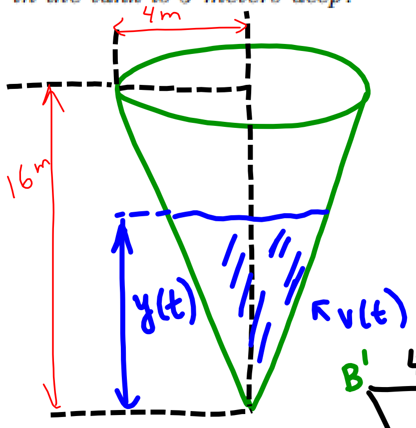
Substitute the given data:

$$\cos\frac{\pi}{4}\,\frac{d\theta}{dt} = \frac{1}{25}\cdot 3$$

$$\frac{1}{\sqrt{2}}\,\frac{d\theta}{dt} = \frac{3}{25}$$

$$\frac{d\theta}{dt} = \frac{3\sqrt{2}}{25}\ \text{rad/s}$$

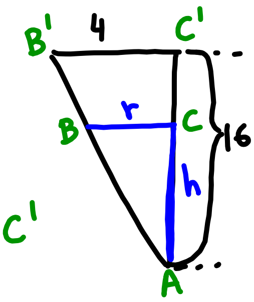
EXAMPLE 3. *A water tank has the shape of an inverted right circular cone with height* 16m *and base radius* 4m. *Water is pouring into the tank at* $3\text{m}^3/\text{min}$. *How fast is the water level rising when the water in the tank is* 5 *meters deep?*



Quantities: $V(t)$, $y(t)$

Given $\frac{dV}{dt} = 3 \ m^3/min$

Find $\frac{dy}{dt}$ when $y = 5$ m.



$$V = \frac{1}{3}\pi r^2 h$$

$\triangle ABC \sim \triangle AB'C'$

$$\frac{B'C'}{BC} = \frac{AC'}{AC}$$

$$\frac{4}{r} = \frac{16}{h} \Rightarrow 4h = 16r$$

$$r = \frac{h}{4}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h$$

$$V = \frac{\pi h^3}{48}$$

$$V(t) = \frac{\pi}{48}\left[y(t)\right]^3$$

Differentiate and use C-R

$$\frac{dV}{dt} = \frac{\pi}{48} 3\left[y(t)\right]^2 \frac{dy}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{16}\left[y(t)\right]^2 \frac{dy}{dt}$$

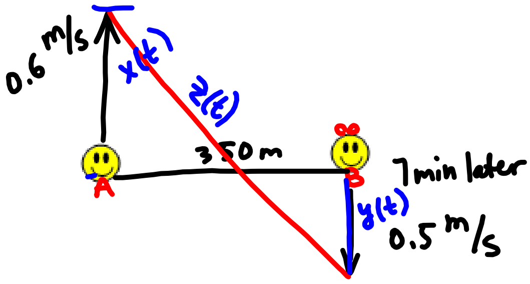
Substitute the given data when $y = 5$:

$$3 = \frac{\pi}{16} 5^2 \frac{dy}{dt}$$

$$3 = \frac{25\pi}{16}\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{48}{25\pi} \ m/min$$

EXAMPLE 4. *Two people are separated by 350 meters. Person A starts walking north at a rate of 0.6 m/sec and 7 minutes later Person B starts walking south at 0.5 m/sec. At what rate is the distance separating the two people changing 25 minutes after Person A starts walking?*



Quantities: $x(t), y(t), z(t)$

Given: $x'(t) = 0.6$ m/sec

$y'(t) = 0.5$ m/sec

Find $z'(t)$ when $t = 25$ min

$[z(t)]^2 = [x(t) + y(t)]^2 + 350^2$

Differentiate:

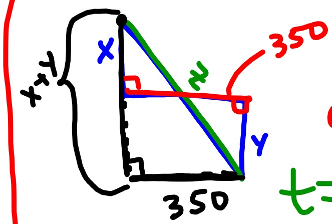
$2z(t)z'(t) = 2(x(t)+y(t))(x'(t)+y'(t))$

When $t = 25$ min:

$2 \cdot 1482 \cdot z' \approx 2(900+540) \times (0.6+0.5)$

$z' \approx 1.07$ m/s

$z^2 = (x+y)^2 + 350^2$



Find $x(t), y(t), z(t)$ after 25 min.

$t = 25$ min $= 25 \cdot 60 = 1500$ s

$x(1500) = $ speed A $\cdot$ time A

$= x' \cdot 1500 = 0.6 \cdot 1500$

$= \boxed{900 \text{ m}}$

$y(1500) = $ Speed B $\cdot$ time B

$= y' \cdot (25-7) \cdot 60$

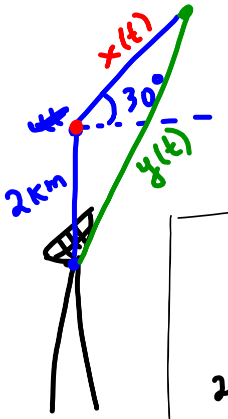
$= 0.5 \cdot 18 \cdot 60 = \boxed{540 \text{m}}$

$z = \sqrt{(x+y)^2 + 350^2}$

$z(1500) = \sqrt{(900+540)^2 + 350^2}$

$\approx \boxed{1482 \text{ m}}$

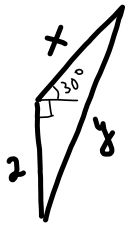
EXAMPLE 5. *A plane flying with a constant speed of 360km/hour passes over a radar station at an altitude of 2km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing 1 minute later?*



Quantities $x(t)$, $y(t)$

Given: $\frac{dx}{dt} = 360 \text{ km/h}$

Find $\frac{dy}{dt}$ when $t = 1\text{min} = \frac{1}{60}h$



Using Law of Cosine:

$y^2 = x^2 + 2^2 - 2\cdot x\cdot 2\cos(30^\circ + 90^\circ)$

$y^2 = x^2 + 4 - 4x\cdot\left(-\frac{1}{2}\right)$

$y^2 = x^2 + 2x + 4$

$[y(t)]^2 = [x(t)]^2 + 2x(t) + 4$

Differentiate both sides:

$\cancel{2}y(t)\,y'(t) = \cancel{2}x(t)\,x'(t) + \cancel{2}x'(t)$

Find $x(t)$ and $y(t)$ after $1\text{min} = \frac{1}{60}h$

$x\left(\frac{1}{60}\right) = x'\cdot\frac{1}{60} = 360\cdot\frac{1}{60} =$

$= 6 \text{ km/h}$

$y\left(\frac{1}{60}\right) = \sqrt{\left[x\left(\frac{1}{60}\right)\right]^2 + 2x\left(\frac{1}{60}\right) + 4}$

$= \sqrt{6^2 + 2\cdot 6 + 4} = \sqrt{52} = \sqrt{4\cdot 13}$

$= 2\sqrt{13} \text{ km/h}$

plug in the numerical data when $t = \frac{1}{60}h$

$2\sqrt{13}\,y'\left(\frac{1}{60}\right) = 6\cdot 360 + 360$

$2\sqrt{13}\,y'\left(\frac{1}{60}\right) = 7\cdot 360$

$y'\left(\frac{1}{60}\right) = \frac{7\cdot 360}{2\sqrt{13}}$

$\approx 350 \text{ km/h}$