3.11: Differentials; Linear Approximations

Differentials: If x changes from x_1 to x_2, then the *change in* x is

$$\Delta x = x_2 - x_1.$$

If $y = f(x)$ then corresponding *change in* y is

$$\Delta y = f(x_2) - f(x_1).$$

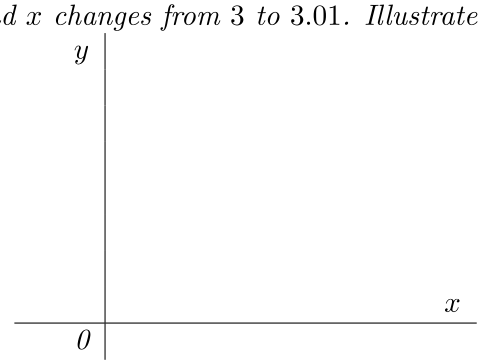
DEFINITION 1. *Let* $y = f(x)$*, where* f *is a differentiable function. Then the differential* $\mathrm{d}x$ *is an independent variable (i.e.* $\mathrm{d}x$ *can be given the value of any real number). The differential* $\mathrm{d}y$ *is then defined in terms of* $\mathrm{d}x$ *by the equation*

$$\mathrm{d}y = f'(x)\mathrm{d}x.$$

EXAMPLE 2. *Compare the values of* Δy *and* $\mathrm{d}y$ *if*

$$y = f(x) = x^2 - 2x$$

and x *changes from* 3 *to* 3.01*. Illustrate these quantities graphically.*



REMARK 3. Notice that $\mathrm{d}y$ was easier to compute than Δy. For more complicated functions (for example $y = \cos x$) it may be impossible to compute Δy exactly.

CONCLUSION: $\Delta y \approx \mathrm{d}y$ provided we keep Δx small.

EXAMPLE 4. *A sphere was measured and its radius was found to be 15 inches with a possible error of no more that 0.02 inches. If we use this value of the radius, find the following quantities:*

(a) *the maximum possible error in the volume of the sphere;*

(b) *the relative error in the radius of the sphere;*

(c) *the relative error in the volume of the sphere.*

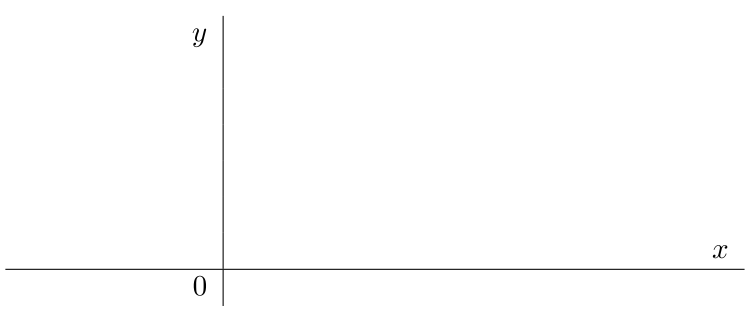
Linear Approximation: The function

$$L_a(x) = f(a) + f'(a)(x - a)$$

(whose graph, $y = L_a(x)$, is the tangent line to the curve $y = f(x)$ at the points $(a, f(a))$) is called the **linearization of f at a.** The approximation $f(x) \approx L_a(x)$ or

$$f(x) \approx f(a) + f'(a)(x - a) \tag{1}$$

is called the **linear approximation** or **tangent line approximation** of f at a.



Rewriting the formula (1) when $x = a + \Delta x$, we get the following "approximation by differentials" formula:

EXAMPLE 5. *Determine the linaerization and the linear approximation for* $\sin x$ *at* $a = 0$. *Then find* $\sin(0.000007)$.

EXAMPLE 6. *Use a linear approximation (or differentials) or to find an approximate value for* $\sin 31°$.

EXAMPLE 7. *Use a linear approximation (or differentials) to find an approximate value for* $\sqrt[4]{0.98}$.