Section 3.1: Derivative

$f'(a)$

$\frac{df(x)}{dx}$

DEFINITION 1. *The* **Derivative** *of a function $f(x)$ at $x = a$ is*

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.}$$

Other common notations for the derivative of $y = f(x)$ are f', $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$.

It follows from the definition that the derivative $f'(a)$ measures:

$m = f'(a)$

- The slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$;
- The instantaneous rate of change of $f(x)$ at $x = a$; $v(a) = f'(a)$
- The instantaneous velocity of the object at time at $t = a$ (if $f(t)$ is the position of an object at time t).

EXAMPLE 2. *Given* $f(x) = \dfrac{3}{x+5}$. *Use definition of the derivative to calculate* $f(x)$ *at* $x = -3$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$a = -3$

$$\frac{df(-3)}{dx} = f'(-3) = \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h\to 0} \frac{\frac{3}{-3+h+5} - \frac{3}{-3+5}}{h}$$

$$= 3\lim_{h\to 0} \frac{\frac{1}{h+2}\cdot\frac{2}{2} - \frac{1}{2}\cdot\frac{h+2}{h+2}}{h} = 3\lim_{h\to 0} \frac{\cancel{2} - (\cancel{h}+\cancel{2})}{2(h+2)\cancel{h}}$$

$$= -\frac{3}{2}\lim_{h\to 0} \frac{1}{h+2} = -\frac{3}{2}\cdot\frac{1}{0+2} = \boxed{-\frac{3}{4}}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

EXAMPLE 3. *Each limit below represents the derivative of function $f(x)$ at $x = a$. State f and a in each case.*

(a) $\lim_{h \to 0} \frac{(3+h)^4 - 81}{h} = \lim_{h \to 0} \frac{(3+h)^4 - 3^4}{h} = \lim_{h \to 0} \frac{f(x+3) - f(3)}{h} = f'(3)$

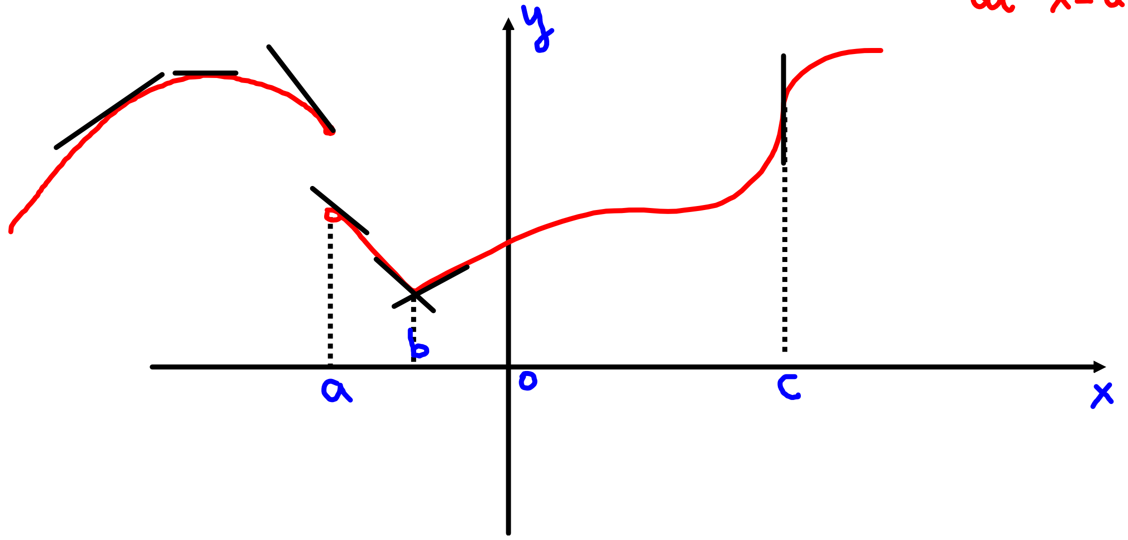
$\boxed{a = 3, \ f(x) = x^4}$

(b) $\lim_{x \to \frac{3\pi}{2}} \frac{\sin x + 1}{x - \frac{3\pi}{2}} = \lim_{x \to \frac{3\pi}{2}} \frac{\sin x - (-1)}{x - \frac{3\pi}{2}} = \lim_{x \to \frac{3\pi}{2}} \frac{\sin x - \sin\frac{3\pi}{2}}{x - \frac{3\pi}{2}} = f'\left(\frac{3\pi}{2}\right)$

$a = \frac{3\pi}{2}, \ f(x) = \sin x$

Question: Where does a derivative not exist for a function?

at $x = a, b, c$



DEFINITION 4. *A function $f(x)$ is said to be* **differentiable** *at $x = a$ if $f'(a)$ exists.*

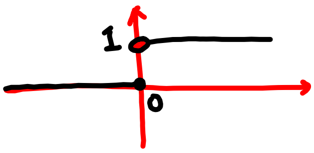
EXAMPLE 5. *Refer to the graph above to determine where f(x) is not differentiable.*

Answer: $x = a, b, c$

CONCLUSION: A function $f(x)$ is NOT differentiable at $x=a$ if

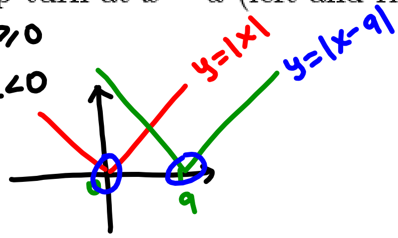
- $f(x)$ is not continuous at $x=a$;

$$f(x)=\begin{cases}1, & x>0\\ 0, & x\le 0\end{cases} \qquad f'(0) \text{ DNE}$$



- $f(x)$ has a sharp turn at $x=a$ (left and right derivatives are not the same);

$$y=|x|=\begin{cases}x, & x\geq 0\\ -x, & x<0\end{cases}$$



$$f(x)=|x-9|$$

$$\boxed{f'(9) \text{ DNE}}$$

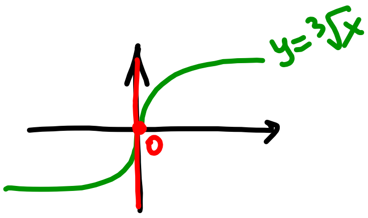
$$f'(9)=\lim_{h\to 0}\frac{f(9+h)-f(9)}{h}=$$

$$=\lim_{h\to 0}\frac{|9+h-9|-|9-9|}{h}=\lim_{h\to 0}\frac{|h|}{h} \text{ DNE, because}$$

$$\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{h}{-h}=\underbrace{-1}_{f_-(9)}\neq\lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=\underset{f_+(9)}{\underset{\shortparallel}{1}}.$$

- $f(x)$ has a vertical tangent at $x=a$.

$$f(x)=\sqrt[3]{x}$$



$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

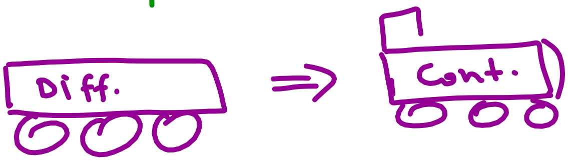
$$=\lim_{h\to 0}\frac{\sqrt[3]{h}-\sqrt[3]{0}}{h}=\lim_{h\to 0}\frac{h^{\frac{1}{3}}}{h^1}$$

$$=\lim_{h\to 0}\frac{1}{h^{2/3}} \text{ DNE}$$

$x=a$

THEOREM 6. *If f is differentiable at a then f is continuous at a.*

$f'(a)$ exists

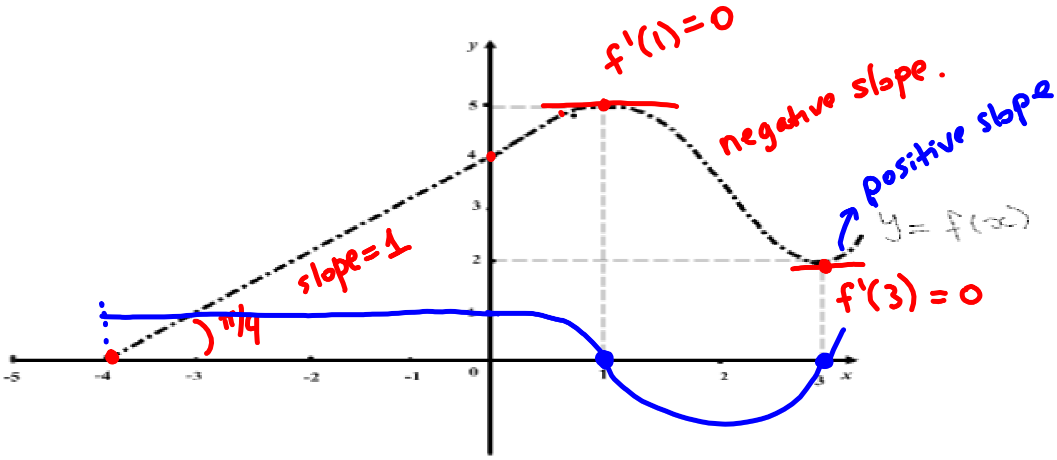


The derivative as a function: If we replace a by x in Definition 1 we get:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

A new function $g(x) = f'(x)$ is called the **derivative of** f.

EXAMPLE 7. *Use the graph of $f(x)$ below to sketch the graph of the derivative $f'(x)$.*



EXAMPLE 7. *Use the definition of the derivative to find $f'(x)$ for $f(x) = \sqrt{1+3x}$.*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{1+3(x+h)} - \sqrt{1+3x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{1+3(x+h)} - \sqrt{1+3x}\right)\left(\sqrt{1+3(x+h)} + \sqrt{1+3x}\right)}{h\left(\sqrt{1+3(x+h)} + \sqrt{1+3x}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{1+3x}+3\cancel{h} - \cancel{(1+3x)}}{\cancel{h}\left(\sqrt{1+3x+3h} + \sqrt{1+3x}\right)} = \lim_{h\to 0} \frac{3}{\sqrt{1+3x+3h}+\sqrt{1+3x}}$$

$$= \frac{3}{\sqrt{1+3x+0}+\sqrt{1+3x}} = \frac{3}{2\sqrt{1+3x}}$$

$$f'(x) = \left(\sqrt{1+3x}\right)' = \frac{d\left(\sqrt{1+3x}\right)}{dx} = \frac{3}{2\sqrt{1+3x}}$$