Section 3.4: Derivatives of Trigonometric Functions

It is important to remember that everything for six trigonometric functions ($\sin x, \cos x, \tan x, \cot x, \csc x, \sec x$ will be done in radians.

EXAMPLE 1. *Compute:*

(a) $\lim_{x\to 0} \sin x = \sin 0 = 0$ (b) $\lim_{x\to 0} \cos x = \cos 0 = 1$

THEOREM 2.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1,} \qquad \boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.}$$

Proof

$$\lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{x\,(\cos x + 1)} = \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$$= -\lim_{x\to 0} \frac{1 - \cos^2 x}{x(\cos x + 1)} = -\lim_{x\to 0} \frac{\sin^2 x}{x(\cos x + 1)}$$

$$= -\lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}$$

$$= -\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{\cos x + 1} = -1 \cdot \frac{0}{1+1} = 0.$$

Corollary: $\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} = \frac{1}{\lim_{x\to 0} \frac{\sin x}{x}} = \frac{1}{1} = 1$

EXAMPLE 3. *Find these limits:*

$\left(u = 5x \xrightarrow[x \to 0]{} 0 \right)$

(a) $\lim_{x\to 0} \frac{5\sin(5x)}{5x} = 5 \lim_{x\to 0} \frac{\sin(5x)}{5x}$

$= 5 \lim_{u\to 0} \frac{\sin u}{u} = 5 \cdot 1 = \boxed{5}$

(c) $\lim_{x\to 0} \frac{x}{\sin(4x)} = \lim_{x\to 0} \frac{1}{\frac{\sin(4x)}{x}} = \frac{1}{\lim_{x\to 0} \frac{\sin 4x}{x}} = \frac{1}{4}$

(b) $\lim_{x\to 0} \frac{\sin(9x)}{\sin(7x)} = \lim_{x\to 0} \frac{x\sin(9x)}{x\sin(7x)} = \lim_{x\to 0} \frac{\sin(9x)}{x} \cdot \frac{x}{\sin(7x)}$

$= \lim_{x\to 0} \frac{\sin(9x)}{x} \cdot \lim_{x\to 0} \frac{x}{\sin(7x)} = 9 \cdot \frac{1}{7} = \frac{9}{7}$

Conclusion: If $a, b \neq 0$ then

$$\lim_{x\to 0} \frac{\sin(ax)}{x} = a, \qquad \lim_{x\to 0} \frac{x}{\sin(ax)} = \frac{1}{a}, \qquad \lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$$

(d) $\lim_{x\to 0}\dfrac{1}{x^2\cot^2(3x)} = \lim_{x\to 0}\dfrac{1}{x^2\,\dfrac{\cos^2(3x)}{\sin^2(3x)}} = \lim_{x\to 0}\dfrac{\sin^2(3x)}{x^2\cos^2(3x)}$

$= \lim_{x\to 0}\left(\dfrac{\sin(3x)}{x}\right)^2\cdot\dfrac{1}{\cos^2(3x)} = \left(\lim_{x\to 0}\dfrac{\sin(3x)}{x}\right)^2\cdot\lim_{x\to 0}\dfrac{1}{\cos^2(3x)}$

$= 3^2\cdot\dfrac{1}{1} = 9$

(e) $\lim_{x\to 0}\dfrac{(\cos x - 1)\,x}{(\sin x)\,x} = \lim_{x\to 0}\dfrac{\cos x-1}{x}\cdot\dfrac{x}{\sin x}$

$= \lim_{x\to 0}\dfrac{\cos x - 1}{x}\cdot\lim_{x\to 0}\dfrac{x}{\sin x} = 0\cdot 1 = 0.$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

EXAMPLE 4. *Find the following derivatives:*

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = (\sin x)' = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$

$$= \lim_{h\to 0} \frac{\sin x \cosh + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin h \cos x + (\sin x \cosh - \sin x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin h \cos x + \sin x(\cosh - 1)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin h \cos x}{h} + \frac{\sin x(\cosh-1)}{h} \right)$$

$$= \cos x \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{=1} + \sin x \underbrace{\lim_{h\to 0} \frac{\cosh - 1}{h}}_{=0} = \cos x.$$

Remark Similarly one can get $(\cos x)' = -\sin x$.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of Trig Functions (memorize these!)

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\cot x = -\csc^2 x$

EXAMPLE 5. *Find the derivative of these functions.*

(a) $y' = \left(\cot x + 5\sec x + x\sqrt{x}\right)'$

$= (\cot x)' + 5(\sec x)' + \left(x^{\frac{3}{2}}\right)'$

$= -\csc^2 x + 5\sec x \tan x + \frac{3}{2}\sqrt{x}$

(b) $f'(x) = \left(\dfrac{\cos x}{1+\sin x}\right)' = \dfrac{(\cos x)'(1+\sin x) - \cos x\,(1+\sin x)'}{(1+\sin x)^2}$

$= \dfrac{-\sin x\,(1+\sin x) - \cos x \cdot \cos x}{(1+\sin x)^2}$

$= \dfrac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \dfrac{-\sin x - \overbrace{(\sin^2 x + \cos^2 x)}^{=1}}{(1+\sin x)^2}$

$= \dfrac{-\sin x - 1}{(1+\sin x)^2} = -\dfrac{\cancel{\sin x + 1}}{(1+\sin x)^{\cancel{2}}} = -\dfrac{1}{1+\sin x}$