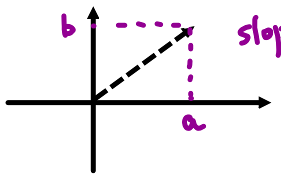


slope of this vector $= \frac{b}{a}$

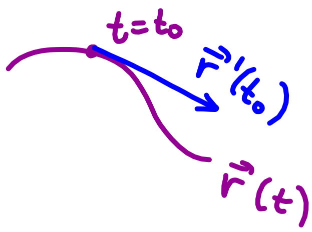
3.9: Slopes and tangents of parametric curves

Consider a curve C given by the parametric equations

$$x = x(t), \quad y = y(t),$$

or in vector form:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle .$$



If both $x(t)$ and $y(t)$ are differentiable, then

$$\mathbf{r}'(t_0) = \langle x'(t_0), y'(t_0) \rangle$$

is a vector that is tangent to C. Its slope is:

$$\text{slope} = \frac{y'(t_0)}{x'(t_0)}$$

Another way to see this is by using the Chain Rule. We have $y = y(x(t))$ and then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

which implies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Remark:

Line through (x_0, y_0) with slope m:

$$y - y_0 = m(x - x_0)$$

Tangent line to the graph of $y = f(x)$ at $x = a$:

$$y - \boxed{f(a)} = \boxed{f'(a)}(x - \boxed{a})$$

Tangent line to the graph of $\vec{r}(t) = \langle x(t), y(t) \rangle$ at $t = t_0$:

$$y - \boxed{y(t_0)} = \boxed{m(t_0)}(x - \boxed{x(t_0)}),$$

where $m(t_0) = \dfrac{y'(t_0)}{x'(t_0)}$.

EXAMPLE 1. *Find the equation of tangent line to the curve*

$$x(t) = \sin t, \quad y(t) = \tan t$$

at the point corresponding to $t = \frac{\pi}{4}$.

Find $\boxed{\text{tangent point}}$ at $t = \frac{\pi}{4}$:

$$\left(x\left(\tfrac{\pi}{4}\right), y\left(\tfrac{\pi}{4}\right)\right) = \left(\sin\tfrac{\pi}{4}, \tan\tfrac{\pi}{4}\right) = \boxed{\left(\tfrac{\sqrt{2}}{2}, 1\right)}$$

Find formula for slope of tangent:

$$m(t) = \frac{y'(t)}{x'(t)} = \frac{(\tan t)'}{(\sin t)'} = \frac{\sec^2 t}{\cos t}$$

$$= \frac{\frac{1}{\cos^2 t}}{\cos t} = \frac{1}{\cos^3 t}$$

Find $\boxed{\text{slope of tangent at } t = \pi/4}$:

$$\boxed{m\left(\tfrac{\pi}{4}\right)} = \frac{1}{\cos^3 \frac{\pi}{4}} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^3} = \left(\sqrt{2}\right)^3$$

$$= \boxed{2\sqrt{2}}$$

Equation of tangent at $t = \pi/4$

$$\boxed{y - 1 = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right)}$$

EXAMPLE 2. *Find the equation of tangent line to the curve*

$$x(t) = t + 1, \quad y(t) = t^2 + 4$$

at the $(2, 5)$. tangent point

<u>Way 1</u> Eliminate parameter to get $y = f(x)$.

<u>Way 2</u> Find t such that

$\langle x(t), y(t) \rangle = \langle 2, 5 \rangle$

$\langle t+1, t^2+4 \rangle = \langle 2, 5 \rangle$

$$\left.\begin{cases} t+1 = 2 & \Rightarrow t = 1 \\ t^2+4 = 5 & \Rightarrow t = \pm 1 \end{cases}\right\} \boxed{t=1}$$

Find formula for slope of tangent

$$m(t) = \frac{y'(t)}{x'(t)} = \frac{(t^2+4)'}{(t+1)'} = \frac{2t}{1} = 2t$$

the slope at tangent point

$$\boxed{m(1) = 2}$$

Equation of tangent line at $(2,5)$:

$$y - 5 = 2(x - 2).$$

EXAMPLE 3. *Find the points on the curve*

$$x = t + t^2, \quad y = t^2 - t$$

where the tangent lines are horizontal and there they are vertical.

$m(t) = \dfrac{y'(t)}{x'(t)}$, $x'(t) = (t+t^2)' = 1+2t$
$y'(t) = (t^2-t)' = 2t-1$

Consider the following cases:

Case 1: $x'(t) = 1+2t = 0 \Rightarrow t = -\frac{1}{2}$
$y'(t) = 2t-1 \neq 0 \Rightarrow t \neq \frac{1}{2}$

$\boxed{\text{tangent line is vertical at}}$ $t = -\frac{1}{2}$:

$$\left(x\left(-\tfrac{1}{2}\right), y\left(-\tfrac{1}{2}\right)\right) = \left(-\tfrac{1}{2} + \left(-\tfrac{1}{2}\right)^2, \left(-\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)\right)$$
$$= \left(-\tfrac{1}{2} + \tfrac{1}{4}, \tfrac{1}{4} + \tfrac{1}{2}\right) = \boxed{\left(-\tfrac{1}{4}, \tfrac{3}{4}\right)}$$

Case 2: $x'(t) = 1+2t \neq 0 \Rightarrow t = \frac{1}{2}$
$y'(t) = 2t-1 = 0$

$\boxed{\text{tangent line is horizontal}}$ at $t = \frac{1}{2}$:

$$\left(x\left(\tfrac{1}{2}\right), y\left(\tfrac{1}{2}\right)\right) = \left(\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2, \left(\tfrac{1}{2}\right)^2 - \tfrac{1}{2}\right)$$
$$= \left(\tfrac{1}{2} + \tfrac{1}{4}, \tfrac{1}{4} - \tfrac{1}{2}\right) = \boxed{\left(\tfrac{3}{4}, -\tfrac{1}{4}\right)}$$

Case 3 $\begin{cases} x'(t) = 1+2t = 0 \\ y'(t) = 2t-1 = 0 \end{cases}$ impossible (no such points)

an additional investigation is usually needed in this case.

REMARK 4. It may happen that $x'(t) = y'(t) = 0$ for some value of t.

Illustration 1. $x(t) = t^3$, $y(t) = t^3$

$x'(t) = 3t^2$
$y'(t) = 3t^2$ $\Rightarrow$ $x'(0) = y'(0) = 0$.

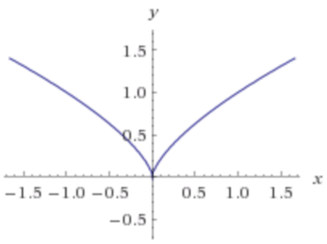
Eliminating parameter, we get $y = x$, i.e. slope at $(0,0)$ is equal 1.

Also, if we reparameterize this curve as $x(t) = t$, $y(t) = t$, we get $\frac{y'(0)}{x'(0)} = \frac{1}{1} = 1$.

Illustration 2. $x(t) = t^3$, $y(t) = t^2$

$x'(t) = 3t^2$, $y'(t) = 2t$
$x'(0) = 0$, $y'(0) = 0$

Eliminating parameter:
$x = t^3$, $y = t^2$
$x^2 = t^6$, $y^3 = t^6$
$x^2 = y^3$



$\boxed{\sin 2t = 2\sin t \cos t}$

EXAMPLE 5. *Show that the curve*

$$x = \cos t, \quad y = \cos t \sin t = \frac{1}{2}\sin 2t$$

has two tangents at $(0, 0)$ *and find their equations.*

First find t such that $(x(t), y(t)) = (0, 0)$:

$$\begin{cases} \cos t = 0 \\ \cos t \sin t = 0 \end{cases} \Rightarrow \cos t = 0$$

because $\cos t$ and $\sin t$ cannot be equal to zero at the same time.

$$\cos t = 0 \Rightarrow t = \frac{\pi}{2} + \pi k \quad (k = 0, \pm 1, \pm 2, \pm 3, \dots)$$

Find a formula for slope of tangent

$$m(t) = \frac{y'(t)}{x'(t)} = \frac{\left(\frac{1}{2}\sin 2t\right)'}{(\cos t)'} = \frac{\frac{1}{\cancel{2}}(\cos 2t)\cdot \cancel{2}}{-\sin t}$$

$$= -\frac{\cos 2t}{\sin t}$$

$\longrightarrow$

Find slope of tangent at (0,0) (or at $t = \frac{\pi}{2} + \pi k$):

$$m\left(\frac{\pi}{2}+\pi k\right) = -\frac{\cos 2\left(\frac{\pi}{2}+\pi k\right)}{\sin\left(\frac{\pi}{2}+\pi k\right)}$$

$$= -\frac{\cos(\pi + 2\pi k)}{\sin\left(\frac{\pi}{2}+\pi k\right)} = -\frac{\cos\pi}{(-1)^k \sin\frac{\pi}{2}}$$

$$= -\frac{-1}{(-1)^k \cdot 1} = \frac{1}{(-1)^k} =$$

$$\begin{cases} \frac{1}{-1} = -1, & \text{if } k \text{ is odd} \\ \frac{1}{1} = 1, & \text{if } k \text{ is even.} \end{cases}$$

So, there are two tangents at (0, 0) with slopes ± 1.

Equations of tangents at (0, 0):

$$\boxed{y = x \quad \text{and} \quad y = -x}$$