4.1: Exponential functions and their derivatives

An **exponential function** is a function of the form

$$f(x) = a^x \qquad \boxed{a>0}$$

where a is a positive constant. It is defined is the following manner:

- If $x = n$, a positive integer, then $a^n = \underbrace{a \cdot a \cdot \ \cdots \ \cdot a}_{n \text{ factors}}$ $\quad 2^8 = \underbrace{2 \cdot 2 \ldots 2}_{8 \text{ times}}$
- If $x = 0$ then $a^0 = 1$.
- If $x = -n$, n is a positive integer, then $a^{-n} = \frac{1}{a^n}$. $\quad 7^{-5} = \frac{1}{7^5}$
- If x is a rational number, $x = \frac{p}{q}$, with p and q integers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}. \qquad 6^{\frac{2}{3}} = \sqrt[3]{6^2}$$

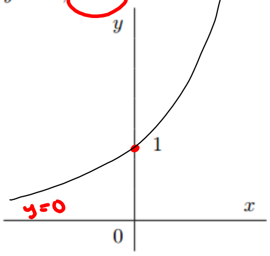
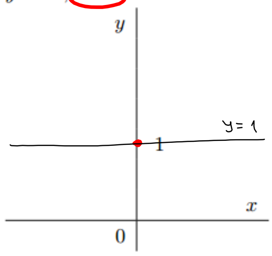
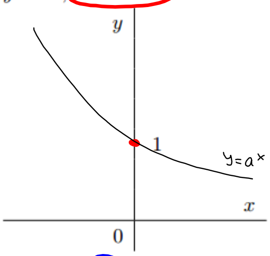
- If x is an irrational number then we define

$$\boxed{a^x = \lim_{r \to x} a^r} \qquad \lim_{x \to 5} 9^x = 9^5$$

  where r is a rational number.

It can be shown that this definition uniquely specifies a^x and makes the function $f(x) = a^x$ continu

There are basically 3 kinds of exponential functions $y = a^x$: $\quad a>0 \quad a^0 = 1 \quad (0,1)$

Exponential growth	Constant	Exponential Decay
$y = a^x$, $a > 1$	$y = 1^x$, $a = 1$	$y = a^x$, $0 < a < 1$
		
Domain: $\mathbb{R}$		Domain: $\mathbb{R}$
Range: $(0,\infty)$		Range: $(0,\infty)$
$\lim_{x\to\infty} a^x = \infty$		$\lim_{x\to\infty} a^x = 0$
$\lim_{x\to-\infty} a^x = 0$		$\lim_{x\to-\infty} a^x = \infty$
horizontal asymptote: $y=0$		horizontal asymptote: $y=0$

PROPERTIES OF THE EXPONENTIAL FUNCTION:

If $a, b > 0$ and x, y are real then

1. $a^{x+y} = a^x a^y$ 2. $a^{x-y} = \frac{a^x}{a^y}$ 3. $(a^x)^y = a^{xy}$ 4. $(ab)^x = a^x b^x$.

$5^{3+4} = 5^3 \cdot 5^4$, $5^{3-4} = \frac{5^3}{5^4}$, $(5^3)^4 = 5^{3\cdot 4} = 5^{12}$, $(5\cdot 7)^3 = 5^3 \cdot 7^3$

EXAMPLE 1. *Find the limit:*

(a) $\lim_{x\to\infty} (4^{-x} - 3) = \lim_{x\to\infty}\left(\frac{1}{4^x} - 3\right) = \lim_{x\to\infty}\left(\left(\frac{1}{4}\right)^x - 3\right) = 0 - 3 = \boxed{-3}$

$0 < \frac{1}{4} < 1 \Rightarrow \left(\frac{1}{4}\right)^x \xrightarrow[x\to\infty]{} 0$

(b) $\lim_{x\to\infty} \left(\frac{\pi}{7}\right)^x = 0$

$0 < \frac{\pi}{7} < 1$

(c) $\lim_{x\to-\infty} (\pi^2 - 7)^x = 0$

$\pi^2 - 7 > 1$

(d) $\lim_{x\to 3^+} \left(\frac{1}{7}\right)^{\frac{x}{x-3}} = \lim_{u\to\infty} \left(\frac{1}{7}\right)^u = 0$

$u = \frac{x}{x-3} \xrightarrow[x\to 3^+]{} +\infty$

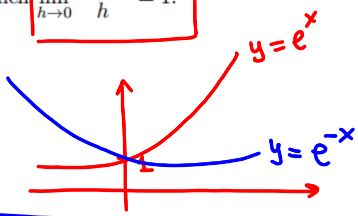
There are in fact a variety of ways to define e. Here are two of them:

1. $\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n}$

2. e is the unique positive number for which $\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1.}$

It can be also shown that $e \approx 2.71828$.

$e > 1$



$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to\infty} e^{-x} = 0$$

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to-\infty} e^{-x} = \infty$$

EXAMPLE 2. *Find the limit:*

(a) $\lim_{x\to 1^+} e^{\frac{4}{x-1}} = \lim_{u\to\infty} e^u = \infty$

$u = \frac{4}{x-1} \xrightarrow[x\to 1^+]{} \infty$

$x > 1$

(b) $\lim_{x\to 1^-} e^{\frac{4}{x-1}} = \lim_{u\to -\infty} e^u = 0$

$u = \frac{4}{x-1} \xrightarrow[x\to 1^-]{} -\infty$

(c) $\lim_{x\to\infty} \frac{e^{5x} - e^{-5x}}{e^{5x} + e^{-5x}} = \lim_{x\to\infty} \frac{e^{5x} - \frac{1}{e^{5x}}}{e^{5x} + \frac{1}{e^{5x}}}$

$\boxed{u = e^{5x} \xrightarrow[x\to\infty]{} \infty}$

$= \lim_{u\to\infty} \frac{u - \frac{1}{u}}{u + \frac{1}{u}} =$

$= \lim_{u\to\infty} \frac{\frac{u^2-1}{\cancel{u}}}{\frac{u^2+1}{\cancel{u}}} = \lim_{u\to\infty} \underbrace{\frac{u^2-1}{u^2+1}}_{\text{rational function}} = \frac{1}{1} = 1.$

Derivative of exponential function.

EXAMPLE 3. *Find the derivative of* $f(x) = e^x$.

$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$

$= \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h} =$

$= e^x \cdot 1 = e^x$

CONCLUSIONS:

1. e^x is differentiable function. and $\frac{d(e^x)}{dx} = (e^x)' = e^x$
2. If $u(x)$ is a differentiable function then by Chain Rule: $\frac{\mathrm{d}}{\mathrm{d}x} e^{u(x)} = e^u \frac{\mathrm{d}u}{\mathrm{d}x}$.

$(e^{u(x)})' = e^{u(x)} \cdot u'(x) = e^u \cdot u'$

EXAMPLE 4. *Find the derivative of the function* $f(x) = e^{x\sin x}$.

$$f'(x) = \left(e^{\overbrace{x\sin x}^{u(x)}}\right)' = e^{x\sin x}(x\sin x)'$$

$$= e^{x\sin x}\left(x'\sin x + x(\sin x)'\right)$$

$$= e^{x\sin x}(\sin x + x\cos x).$$

EXAMPLE 5. *For what value(s) of* A *does the function* $y = e^{Ax}$ *satisfy the equation* $\underbrace{y'' + 2y' - 8y = 0}_{\text{differential equation}}$?

$$y = e^{Ax}$$
$$y' = Ae^{Ax}$$
$$y'' = A^2e^{Ax}$$

$$y'' = A^2e^{Ax}$$
$$+\quad 2y' = 2Ae^{Ax}$$
$$-8y = -8e^{Ax}$$

$$\underbrace{0 = \underbrace{e^{Ax}}_{\neq 0}(A^2+2A-8)}$$

$$\Downarrow$$

$$A^2 + 2A - 8 = 0$$
$$(A+4)(A-2) = 0$$
$$\boxed{A = -4, \quad A = 2.}$$

Conclusion:
The functions $y(x) = e^{-4x}$ and $y(x) = e^{2x}$ are solutions of the given differential equation.