4.3:Logarithmic Functions

DEFINITION 1. *The exponential function* $f(x) = a^x$ *with* $a \neq 1$ *is a one-to-one function. The inverse of this function, called the* **logarithmic function** *with* **base** a*, is denoted by* $f^{-1}(x) = \log_a x$.

Namely,

$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x.$$

In other words, if $x > 0$ then $\log_a(x)$ is the exponent to which the base a must be raised to give x.

EXAMPLE 2. *Evaluate*

(a) $\log_2 16$

(b) $\log_3 \frac{1}{81}$

(c) $\log_{125} 5$

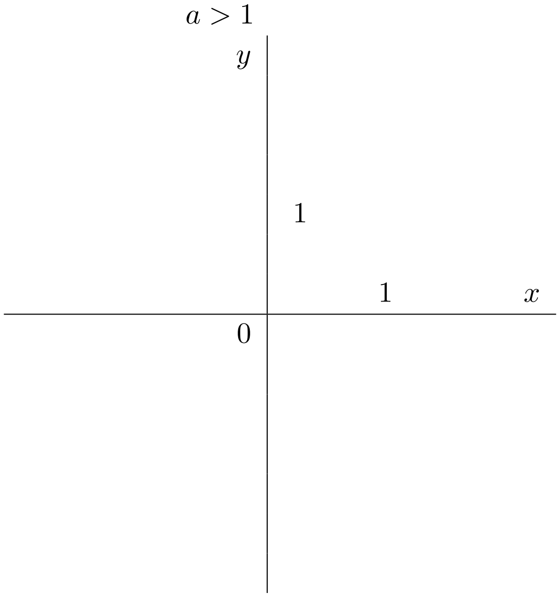
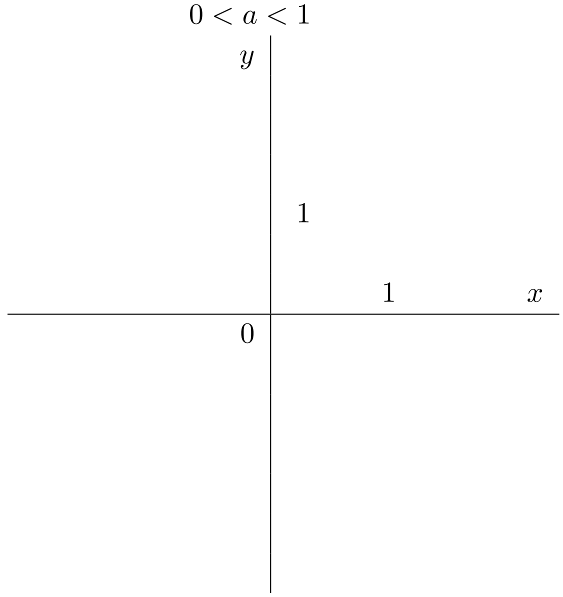
(d) $\log_1 125$

(e) $\log_a 1$

CANCELLATION RULES:

- $\log_a a^x = x$ for all $x \in \mathbb{R}$
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of logarithmic functions $y = \log_a x$:

$a > 1$	$0 < a < 1$
	
Domain:	Domain:
Range:	Range:
$\lim_{x\to\infty} \log_a x =$	$\lim_{x\to\infty} \log_a x =$
$\lim_{x\to 0^+} \log_a x =$	$\lim_{x\to 0^+} \log_a x =$
Vertical asymptote:	Vertical asymptote:
Horizontal asymptote:	Horizontal asymptote:

Properties: Assume that $a \neq 1$ and $x, y > 0$.

$$\log_a(xy) = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^y) = y \log_a x$$

In particular,

$$\log_a \sqrt{x} =$$
$$\log_a \sqrt[n]{x} =$$

Notation: *Common Logarithm:* $\log x = \log_{10} x$. (Thus, $\log x = y \quad \Leftrightarrow \quad 10^y = x$.)
Natural Logarithm: $\ln(x) = \log_e(x)$. (Thus, $\ln x = y \quad \Leftrightarrow \quad e^y = x$.)

Properties of the natural logarithms:

- $\ln(e^x) =$
- $e^{\ln x} =$
- $\ln e =$
- $\log_a x = \dfrac{\ln x}{\ln a}$, where $a > 0$ and $a \neq 1$;
- $\lim_{x\to\infty} \ln x =$
- $\lim_{x\to 0^+} \ln x =$

EXAMPLE 3. *Find each limit:*

(a) $\displaystyle\lim_{x\to\infty} \ln(x^2 - x) =$

(b) $\displaystyle\lim_{x\to 0^+} \log(\sin x) =$

EXAMPLE 4. *Find the domain of $f(x) = \ln(x^3 - x)$.*

EXAMPLE 5. *Solve the following equations:*

(a) $\log_{0.5}(\log(x+120)) = -1$

(b) $e^{5+2x} = 4$

(c) $\log(x-1) + \log(x+1) = \log 15$

EXAMPLE 6. *Find the inverse of the following functions:*

(a) $f(x) = \ln(x + 12)$

(b) $f(x) = \dfrac{10^x - 1}{10^x + 1}$

Change of Base formula:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

In particular,

$$\log_a x = \frac{\ln x}{\ln a}.$$

EXAMPLE 7. *Using calculator and the change-of-base formula evaluate* $\log_2 15$ *to four decimal places.*

Solution.

$$\log_2 15 = \frac{\ln 15}{\ln 2} \approx 3.9069$$