6.1: Sigma notation

DEFINITION 1. *If* $a_m, a_{m+1}, a_{m+2}, ..., a_n$ *are real numbers and* m *and* n *are integers such that* $m \leq n$*, then*

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \ldots + a_n.$$

EXAMPLE 2. *Compute the summation*

$$\sum_{i=1}^{4} \frac{(-1)^k}{k}$$

EXAMPLE 3. *Write the sum in sigma notation:*

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$$

THEOREM 4. *If* c *is any constant then*

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$$

$$\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$$

Note that in general

$$\sum_{i=m}^{n} a_i b_i \neq \left(\sum_{i=m}^{n} a_i\right) \cdot \left(\sum_{i=m}^{n} b_i\right).$$

THEOREM 5.

- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} c = nc$, *where* c *is a constant.*
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 6. *Compute these sums:*

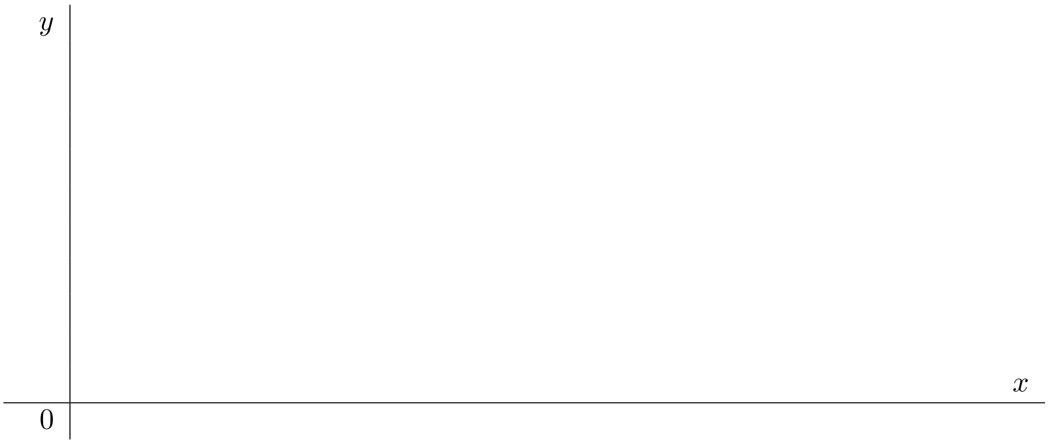
(a) $\displaystyle\sum_{i=1}^{n} i(i+4) =$

(b) $\displaystyle\sum_{j=1}^{n} \left[\left(\frac{j}{n}\right)^3 + 1\right] =$

EXAMPLE 7. *Find the limit:* $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} \left[\left(\frac{j}{n}\right)^3 + 1\right]$

6.2: Area

Area problem: Let a function $f(x)$ be positive on some interval $[a, b]$. Determine the area of the region S between the function and the x-axis.



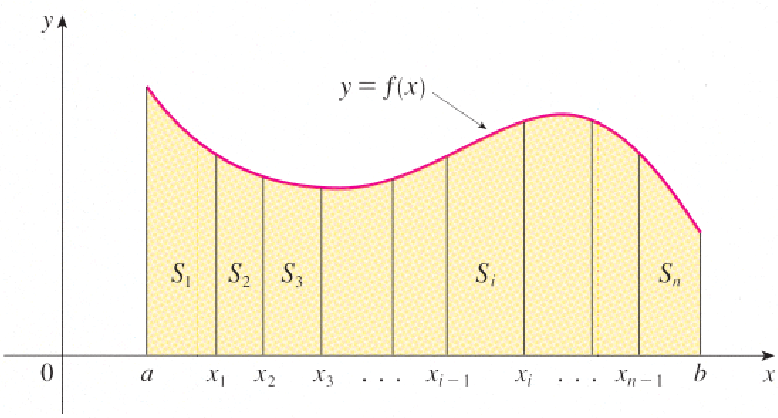
Solution: Choose **partition** points $x_0, x_2, \ldots, x_{n-1}, x_n$ so that

$$a = x_0 \leq x_2 \leq \ldots \leq x_{n-1} \leq x_n = b.$$

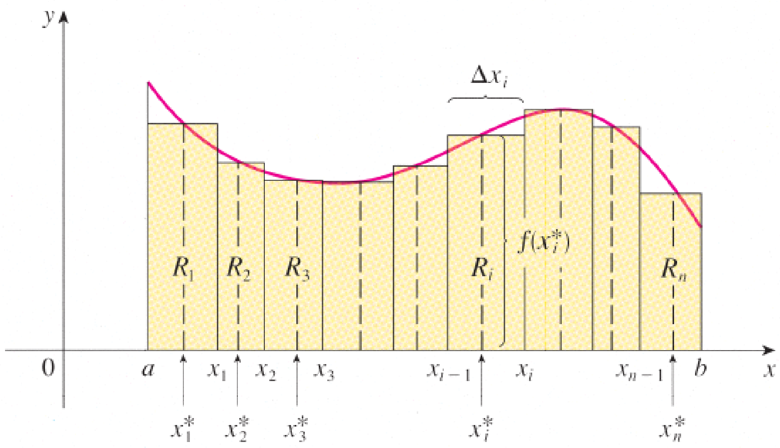
Use notation $\Delta x_i = x_i - x_{i-1}$ for the length of ith subinterval $[x_{i-1}, x_i]$ $(1 \leq 1 \leq n)$

The length of the longest subinterval is denoted by $\|P\|$.

Use the partition P to divide the region S into strips $S_1, S_2, \ldots, S_n$.



Approximate the strips $S_1, S_2, \dots, S_n$ by rectangles $R_1, R_2, \dots, R_n$.



The location in each subinterval where we compute the height is denoted by x_i^*.

The area of the ith rectangle is

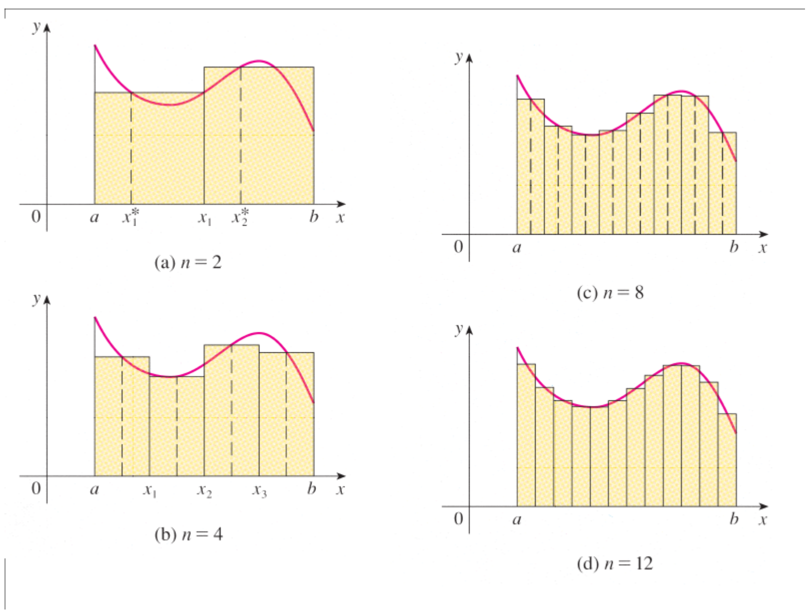
$$A_i =$$

Then

$$A \approx$$

The area A of the region is:

$$A =$$

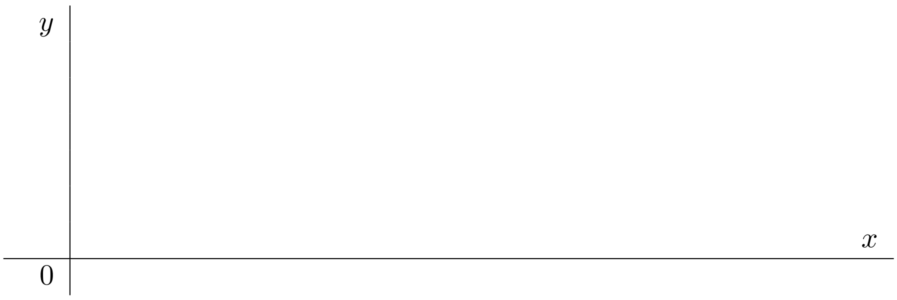


Riemann Sum for a function $f(x)$ on the interval $[a,b]$ is a sum of the form:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Consider a partition has equal subintervals: $x_i = a + i\Delta x$, where $\Delta x = \dfrac{b-a}{n}$.

LEFT-HAND RIEMANN SUM : $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f(a+(i-1)\Delta x)\Delta x$



RIGHT-HAND RIEMANN SUM : $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$

MIDPOINT RIEMANN SUM : $M_n = \sum_{i=1}^{n} f\left(\dfrac{x_i + x_{i-1}}{2}\right)\Delta x$

EXAMPLE 8. *Given* $f(x) = x^3 + 1$ *on* $[0, 1]$.

(a) *Calculate* L_2, R_2, M_2.

(b) *Represent area bounded by* $f(x) = x^3 + 1$ *on the interval* $[0, 1]$ *using right endpoints by Riemann sum.*

(c) *Find the area bounded by* $f(x) = x^3 + 1$ *on the interval* $[0, 1]$.

6.3: The Definite Integral

DEFINITION 9. *The* **definite integral of** f **from** a **to** b *is*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

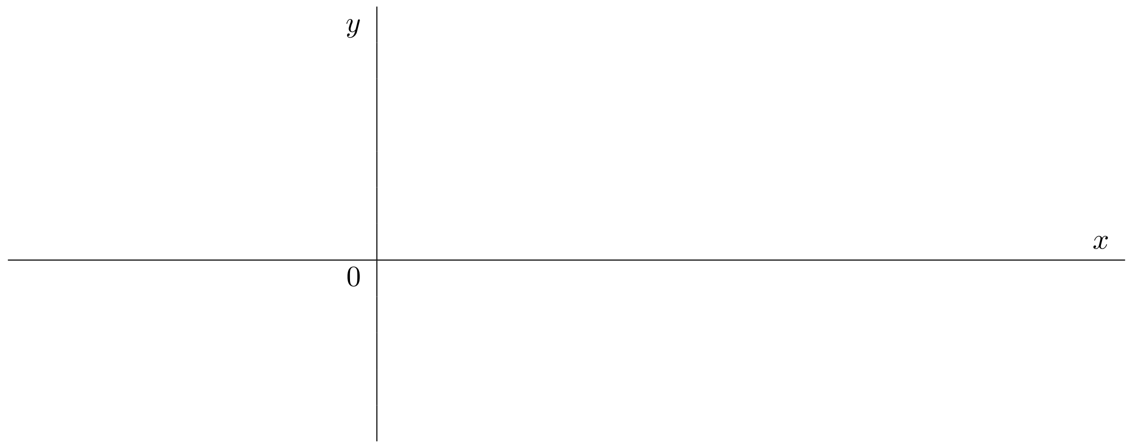
if this limit exists. If the limit does exist, then f *is called* **integrable** *on the interval* $[a, b]$.

If $f(x) > 0$ on the interval $[a, b]$, then the definite integral is the area bounded by the function f and the lines $y = 0$, $x = a$ and $x = b$.

In general, a definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x)\,\mathrm{d}x = A_1 - A_2$$

where A_1 is the area of the region above the x and below the graph of f and A_2 is the area of the region below the x and above the graph of f.



EXAMPLE 10. *Evaluate the integrals by* $\int_{-1}^{3} (2 - x)\,\mathrm{d}x$ *interpreting it in terms of areas.*

Properties of Definite Integrals:

- $\int_a^b \mathrm{d}x = b - a$
- $\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$, where c is any constant
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, where $a \le c \le b$
- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\int_a^a f(x)\,\mathrm{d}x = 0$
- If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge 0$
- If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge \int_a^b g(x)\,\mathrm{d}x$
- If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$.

EXAMPLE 11. *Write as a single integral:*
$\int_3^5 f(x)\,\mathrm{d}x + \int_0^3 f(x)\,\mathrm{d}x - \int_6^5 f(x)\,\mathrm{d}x + \int_5^5 f(x)\,\mathrm{d}x$

EXAMPLE 12. *Estimate the value of* $\int_0^\pi (4\sin^5 x + 3)\,\mathrm{d}x$

6.4: The fundamental Theorem of Calculus

The fundamental Theorem of Calculus :

PART I If $f(x)$ is continuous on $[a,b]$ then $g(x)=\int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x)=f(x)$.

EXAMPLE 13. *Differentiate* $g(x)=\int_{-4}^{x} e^{2t}\cos^2(1-5t)\,\mathrm{d}t$

THEOREM 14. *Let $u(x)$ be a differentiable function and $f(x)$ be a continuous one. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{u(x)} f(t)\,\mathrm{d}t\right)=f(u(x))u'(x).$$

EXAMPLE 15. *Differentiate* $g(x)=\int_{-4}^{x^3} e^{2t}\cos^2(1-5t)\,\mathrm{d}t.$

PART II If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative fort $f(x)$ then

$$\int_a^b f(x)\,\mathrm{d}x=F(x)\Big|_a^b=F(b)-F(a).$$

EXAMPLE 16. *Evaluate*

(a) $\int_{-\pi/2}^{0}(\cos x-4\sin x)\,\mathrm{d}x$

(b) $\int_1^5 \frac{1}{x^2}\,\mathrm{d}x$

(c) $\int_0^1 (u^3+2)^2\,\mathrm{d}u$