1.1 Language and Logic

Mathematical Statements

DEFINITION 1. *A **proposition** is any declarative sentence (i.e. it has both a subject and a verb) that is either true or false, but not both.*

A proposition cannot be neither true nor false and it cannot be both true and false.
A proposition is an example of mathematical statement. Sometimes it is called a statement.

EXAMPLE 2. *Determine whether the following sentences are propositions.*

1. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The integer 5 is odd.*
2. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The integer 24277151704311 is prime.*
3. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ $20 + 19 = 2019$
4. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *Substitute the number* 7 *for* x.
5. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *What is the derivative of* $\cos x$*?*
6. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *Apple manufactures computers.*
7. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The* 2019*th digit of* π *is 5.*
8. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *Vanilla is better than chocolate.*
9. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *I am telling a lie.*

SET Terminology and Notation (very short introduction[1])
Set is a well-defined collection of objects.
Elements are objects or members of the set.

- **Roster notation:**
  $A = \{a, b, c, d, e\}$ Read: Set A with elements a, b, c, d, e.
- **Indicating a pattern:**
  $B = \{a, b, c, ..., z\}$ Read: Set B with elements being the letters of the alphabet.

If a is an element of a set A, we write $a \in A$ that read "a belongs to A." However, if a does not belong to A, we write $a \notin A$.

Some Number sets:

- $\mathbb{R}$ is the set of all *real* numbers
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, the set of all *integers*
- $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of all *natural* numbers
- $\mathbb{Q}$ is the set of all *rational numbers*

[1] We will study SETS in Chapter 4!

- $\mathbb{E}$ is the set of all *even integers*
- $\mathbb{O}$ is the set of all *odd integers*
- $n\mathbb{Z}$ is the set of all integers multiples of n ($n \in \mathbb{Z}$)

EXAMPLE 3. *Determine whether the following sentences are propositions.*

1. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ $|x| > 7$
2. $\boxed{\text{YES}}$ / $\boxed{\text{NO}}$ *The absolute value of a real number x is greater than 7.*

*A **predicate** is any declarative sentence containing one or more variables, each variable representing a value in some prescribing set, called the **universe**, and which becomes a proposition when values from their respective universes are substituted for these variables.*

A predicate is another example of mathematical statement. Sometimes it is also called an open sentence.

Emphasize that all variables in a predicate are *free variables*, i.e. variables that wee need to "substite for" in order to obtain a proposition.

EXAMPLE 4. **(a)** *"$P(x) : x + 5 = 7$ where $x \in \mathbb{R}$" is a predicate.*

- *$P(2)$ is* ________________________________
- *$P(3)$, $P(-1)$, $P(5.6)$ are* ________________________________.
- *$P(x)$ becomes a true proposition when we substitute for x the values from the set* ____________________
  For all other values of x, $P(x)$ is a ________________ *proposition.*

(b) *"$P(n) : (n - 3)^2 \leq 1$ where $n \in \mathbb{Z}$" is a predicate.*

- *$P(n)$ becomes a true proposition when we substitute for n the values from the set* ____________________
  For all other values of n, $P(n)$ is a ________________ *proposition.*

Basic Logical Connectivities

We have two types of mathematical statements: propositions and predicates. We can build more complicated (compound) statements using the following logical connectivities:

$$\wedge, \quad \vee, \quad \neg, \quad \Rightarrow .$$

Conjunction and Disjunction

Logical connectivity	write	read	meaning
Conjunction	$P \wedge Q$	P and Q	Both P and Q are true
Disjunction	$P \vee Q$	P or Q	P is true or Q is true

P: Ben is a student.
Q: Ben is a grader.
$P \wedge Q$: Ben is a students ______a grader.
$P \vee Q$: Ben is a students ______a grader.

TRUTH TABLES

P	Q	$P \wedge Q$	$Q \wedge P$	$P \vee Q$	$Q \vee P$

EXAMPLE 5. *Rewrite the following predicates (over $\mathbb{R}$) using disjunction or conjunction.*

(a) $P(x): \quad |x| \geq 10.$

(b) $P(x): \quad |x| < 10.$

NEGATION

DEFINITION 6. *If P is a mathematical statement, then the* **negation/denial** *of P, written $\neg P$ (read "not P"), is the mathematical statement "P is false".*

TRUTH TABLE

P	$\neg P$
T	
F	

Although $\neg P$ could always be expressed as

It is not the case that P.

there are usually better (useful) ways to express the statement $\neg P$.

`If a statement is true, then its negation is false (and if a statement is false, then its negation is true).`

REMARK 7. Often, a more useful way to express the negated statement is to express it "positively", if possible. The following statements might be useful for that.

FACT *Every integer is either even, or odd.*

Trichotomy Axiom *Given fixed real numbers a and b, exactly one of the statements $a < b$, $a = b$, $b < a$ is true.*

EXAMPLE 8. *Negate the following propositions in a useful way.*

1. P : *The integer 7 is even.*

   ____

   ____

   ____

2. $P: \quad 5^3 = 120 \qquad \neg P:$________________

Implications

DEFINITION 9. *Let P and Q be statements. The* **implication** $P \Rightarrow Q$ *(read "P implies Q") is the statement "If P is true, then Q is true."*

In implication $P \Rightarrow Q$, P is called *assumption, or hypothesis, or antecedent;* and Q is called *conclusion or consequent.*

EXAMPLE 10. *If you score 90% or above in this class, then you get A.*

The truth table for implication:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

REMARK 11. `A false statement implies anything.`

Alternative Expressions for $P \Rightarrow Q$.

If P, then Q.	P implies Q.	P only if Q.	P is sufficient for Q.
	Q if P.	Q when P.	Q is necessary for P.
	Q whenever P.	Q, provided that P.	Whenever P, then also Q.

Converse and Contrapositive

DEFINITION 12. *The statement $Q \Rightarrow P$ is called a* **converse** *of the statement $P \Rightarrow Q$.*

DEFINITION 13. *The statement $(\neg Q) \Rightarrow (\neg P)$ is called a* **contrapositive** *of the statement $P \Rightarrow Q$.*

EXAMPLE 14.

Statement S	$(x = 2) \Rightarrow (x + 1 = 3)$	*True/ False*
Converse of S		*True/ False*
Contrapositive of S		*True/ False*

EXAMPLE 15.

Statement S	$(x = 2) \Rightarrow (x^2 - 4 = 0)$	*True/ False*
Converse of S		*True/ False*
Contrapositive of S		*True/ False*

Biconditional "$\Leftrightarrow$"

For statements P and Q,

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. The biconditional $P \Leftrightarrow Q$ is stated as

"*P is equivalent to Q.*" or "*P if and only if Q.*" (or "*P iff Q.*")
or as "*P is a necessary and sufficient condition for Q.*"

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T			
T	F			
F	T			
F	F			

EXAMPLE 16. *Complete:*

(a) *The biconditional* "The number 17 is odd if and only if 57 is prime." *is* ____________.

(a) *The biconditional* "The number 24 is even if and only if 17 is prime." *is* ____________.

(a) *The biconditional* "The number 17 is even if and only if 24 is prime." *is* ____________.

Logical Equivalence

DEFINITION 17. *Two compound statements are* **logically equivalent** *(write "$\equiv$") if they have the same truth tables, which means they both are true or both are false.*

Question: Are the statements $P \Rightarrow Q$ and $Q \Rightarrow P$ *logically equivalent?* __________

Some Fundamental Properties of Logical Equivalence

THEOREM 18. *Let P, Q and R be statement forms. Then*

1. *Double Negation Law*
   $\neg(\neg P) \equiv P$

2. *Idempotent Laws*
   $P \vee P \equiv P$
   $P \wedge P \equiv P$

3. *Commutative Laws*
   $P \vee Q \equiv$
   $P \wedge Q \equiv$

4. *Associative Laws*
   $P \vee (Q \vee R) \equiv$
   $P \wedge (Q \wedge R) \equiv$

5. *Distributive Laws*
   $P \vee (Q \wedge R) \equiv$
   $P \wedge (Q \vee R) \equiv$

6. $P \vee (Q \vee R) \equiv (P \vee Q) \vee (P \vee R)$
   $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge (P \wedge R)$

7. *De Morgan's Laws*
   $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
   $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

Proof. Each part of the theorem is verified by means of a truth table or/and by a deductive proof.

P	Q		

P	Q			

THEOREM 19. *Let P and Q be statement forms. Then*

(a) $\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q)$.

(b) $P \Rightarrow Q \equiv (\neg P) \vee Q$

(c) $P \Rightarrow Q \equiv (\neg Q) \Rightarrow (\neg P)$ *(i.e. every negation is logically equivalent to its contrapositive!)*

(d) $P \Rightarrow Q$ *is NOT logically equivalent to* $Q \Rightarrow P$

Proof.

(a) Method 1: use truth tables

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
T	T		
T	F		
F	T		
F	F		

P	Q	$\neg Q$	$P \wedge (\neg Q)$
T	T		
T	F		
F	T		
F	F		

Method 2: deductive proof

Note that $P \Rightarrow Q$ is false only if ____________________.

Thus $\neg(P \Rightarrow Q)$ is true only if ____________________.

On the other hand, $P \wedge (\neg Q)$ is true only if ____________________ or only if ____________________.

(b)

(c) Use truth tables.

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$

(d)

COROLLARY 20. $P \Rightarrow Q \equiv (\neg Q) \Leftrightarrow (\neg P)$

EXAMPLE 21. *Determine if the following statements are logically equivalent. Then negate all of them.*

	STATEMENT	*NEGATION*
	If the integer 5 is even, then 5 is prime. *The integer 5 is not even or 5 is prime.* *If the integer 5 is not prime, then 5 is not even.*	

Tautologies and Contradictions

Tautology ($\top$) is a statement that is always true.
Contradiction ($\perp$) is a statement that is always false

P	$\neg P$	$P \vee (\neg P)$	$P \wedge (\neg P)$
T			
F			

Methods to verify tautology/contradiction: truth table and deductive proof.

THEOREM 22. *Let P be a statement form. Then*

(a) *Identity Laws*
$P \wedge \top \equiv P$
$P \vee \perp \equiv P$

(b) *Domination Laws*
$P \vee \top \equiv \top$
$P \wedge \perp \equiv \perp$

(a) *Negation Laws*
$P \vee (\neg P) \equiv \top$
$P \wedge (\neg P) \equiv \perp$

REMARK 23. Let P and Q be statements. The biconditional $P \Leftrightarrow Q$ is a tautology if and only if P and Q are logically equivalent. So, the statements of Theorems 18 and 19 can be rewritten using tautologies. For example, by Theorem 19 (a), we obtain that the statement $\neg(P \Rightarrow Q) \Leftrightarrow P \wedge (\neg Q)$ is a tautology, i.e.

$$(\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge (\neg Q))) \equiv \top.$$

Quantified Statements

EXAMPLE 24. *Consider the following* predicate *over* $\mathbb{N}$:

$$P(n): \ 4n+3 \text{ is prime.}$$

How to convert this predicate into a proposition with a truth value?

A predicate can be made into a proposition by using **quantifiers**.
Universal: $\forall x$ means for all/for every assigned value a of x.
Existential: $\exists x$ means that for some assigned values a of x.

Quantified statement in symbols	Quantified statement in words
"$\forall x \in D, P(x)$." or "$(\forall x \in D) P(x)$."	"For every $x \in D$, $P(x)$." "If $x \in D$, then $P(x)$."
"$\exists x \in D \ni P(x)$" or "$(\exists x \in D) P(x)$"	"There exists x such that $P(x)$."

Once a quantifier is applied to a variable, then the variable is called a **bound** variable. The variable that is not bound is called a **free** variable.

Expressions that can be used in place of for all are

for every, for arbitrary, for any, for each, and given any.

Expression that can be used in place of their exists are

there is a, we can find a, there is at least one, for some, and for at least one.

EXAMPLE 25. *For each of the following propositions, determine if it has any universal or existential quantifiers. If it has universal quantifiers, rewrite it in the form "for all …". If it has existential quantifiers, rewrite it in the form "there exists … such that …".*

1. *The area of a rectangle is its length times its width.*
   Quantifiers:

2. *A triangle may be equilateral.*
   Quantifiers:

3. $15 - 5 = 10$
   Quantifiers:

EXAMPLE 26. *Rewrite the following statements in symbols. Introduce variables, where appropriate.*

a) *The formula $x + 5 = 7$ holds for some real number x.*

b) *For every integer n, either $n \leq 1$ or $n^2 \geq 4$.*

c) *The sum of an even integer and an odd integer is even.*

d) *All positive real numbers have a square root.* (*Do not use symbol* $\sqrt{\ }$.)

Universal Conditional Propositions

The proposition of the form

$$\forall x \in D, P(x) \tag{1}$$

can be rewritten as

$$\forall x, (x \in D) \Rightarrow P(x). \tag{2}$$

If, in addition, there is a set A containing the set D, then (1) and (2) can be expressed as

$$\forall x \in A, (x \in D) \Rightarrow P(x). \tag{3}$$

We will say that the propositions (2) and (3) are *conditional forms* of the universal proposition (1).

EXAMPLE 27. *Express the proposition*

Every octagon has eight sides.

in a conditional form.

EXAMPLE 28. *Express the proposition*

Between any two real numbers there is a rational number.

in a conditional form in two different ways.

Converse and contrapositive of universal conditional propositions

Consider the proposition

$$\forall x \in D, P(x) \Rightarrow Q(x). \tag{4}$$

Applying Definitions 12 and 13, we, respectively, obtain

The converse of (4): $\forall x \in D, Q(x) \Rightarrow P(x)$.

Keep in mind Theorem 19(d).

The contrapositive of (4): $\forall x \in D, (\neg Q(x)) \Rightarrow (\neg P(x))$.

Keep in mind Theorem 19(c).

EXAMPLE 29. *Rewrite the statement*

If m and n are odd integers then $m + n$ is even.

in symbols. Then write its contrapositive and converse both in symbols and words.

NEGATIONS of quantified statements

Rules to negate quantified propositions

	Negation
$\forall x \in D, P(x)$	$\exists x \in D \ni (\neg P(x))$
$\exists x \in D \ni P(x)$	$\forall x \in D, (\neg P(x))$

EXAMPLE 30. *Negate the following*

$\exists x \in D \ni (P(x) \wedge Q(x))$	
$\forall x \in D, (P(x) \Rightarrow Q(x))$	

REMARK 31. Often, a more useful way to express the negated statement is to express it "positively", if possible. Using Trichotomy Axiom, DeMorgan's Law etc is useful. Also if the statement is quantified, the quantifier must be identified and replaced accordingly in order to get a useful denial.

EXAMPLE 32. *Write a useful negation*

1. *S: All continuous functions are differentiable.*

   $\neg S$:____

2. *S : A triangle may be equilateral.*

   $\neg S$:____

EXAMPLE 33. *The goal of this exercise is to obtain a useful negation of the given statement S. The following steps might be helpful.*

1. *Rewrite S in symbols using quantifiers (it might be helpful to introduce temporary variables).*
2. *Express the negation of S in symbols using the above rules.*
3. *Express $\neg S$ in words (try to use the same wording as in original statement).*

a) *S: There exists a prime number p which is greater than 7 and less than 10.*

S____

$\neg S$____

$\neg S$____

b) *P : For every even integer n there exists an integer m such that $n = 2m$.*

S____

$\neg S$____

$\neg S$____

c) P : *If n is an odd integer then $3n+7$ is odd.*

S____

$\neg S$____

$\neg S$____

d) S: *If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.*

S____

$\neg S$____

$\neg S$____

____

Alternative Expressions for $P \Rightarrow Q$. Necessary and Sufficient Conditions

If P, then Q.	P implies Q.	P only if Q.	P is *sufficient* for Q.
	Q if P.	Q when P.	Q is *necessary* for P.
	Q whenever P.	Q, provided that P.	Whenever P, then also Q.

- A condition Q is said to be **necessary** for a condition P, if (and only if) the falsity of Q guarantees the falsity of P.
- A condition P is said to be **sufficient** for a condition Q, if (and only if) the truth of P guarantees the truth of Q.

EXAMPLE 34. *Consider the following predicates*
$P(x)$: *x is a multiple of 4.* $Q(x)$: *x is even. Complete:*

- *"For every integer integer $x, P(x) \Rightarrow Q(x)$" is* ____________________.
- *$P(x)$ is a* ____________________ *condition for Q to be true.*
- *$Q(x)$ is a* ____________________ *condition for $P(x)$ to be true.*
- *$Q(x)$ is not a* ____________________ *condition for $P(x)$ to be true.*

EXAMPLE 35. *Consider the following predicates*
$P(f)$: f is a differentiable function.
$Q(f)$: f is a continuous function.
Complete:

- *"For every real-valued function $f, P(f) \Rightarrow Q(f)$" is* ____________________.
- *"For every real-valued function $f, Q(f) \Rightarrow P(f)$" is* ____________________.
- *$Q(f)$ is a* ____________________ *condition for f to be differentiable, but not a* ____________________ *condition.*
- *$P(f)$ is a* ____________________ *condition for f to be continuous.*

1.2&2.1 Proof

Logical arguments

Most theorems (or results) are stated as implications.

Trivial and Vacuous Proofs[2]

Let $P(x)$ and $Q(x)$ be predicates over a domain D. Consider the quantified statement $\forall x \in D, P(x) \Rightarrow Q(x)$, i.e.

$$\textit{For } x \in D, \textit{ if } P(x) \textit{ then } Q(x). \qquad (\#)$$

or

$$\textit{Let } x \in D. \textit{ If } P(x), \textit{ then } Q(x).$$

`The truth table for implication` $P(x) \Rightarrow Q(x)$ `for an arbitrary (but fixed) element` $x \in D$`:`

$P(x)$	$Q(x)$	$P(x) \Rightarrow Q(x)$
T	T	T
T	F	F
F	T	T
F	F	T

Trivial Proof If it can be shown that $Q(x)$ is true for all $x \in D$ (regardless the truth value of $P(x)$), then (#) is true (according the truth table for implications).

Vacuous Proof If it can be shown that $P(x)$ is false for all $x \in D$ (regardless of the truth value of $Q(x)$), then (#) is true (according the truth table for implications).

EXAMPLE 36. *Determine the truth or falsehood of the following statements.*

(a) For every real x, if $x^2 < 0$, then $x^6 - 3x^4 + x + 3 < 0$.

(b) For every real x, if $x^6 - 3x^4 + x + 3 < 0$, then $e^x > 0$.

(c) If a real valued function is differentiable at some point, then it is also continuous there.

[2]These kind of proofs are rarely encountered in mathematics, however, we consider them as important reminders of implications.

Terminology

Axiom is a true mathematical statement whose truth is accepted without proof.

Definition is an assignment of language and syntax to some property of a set, function, or other object. A definition is not something you prove, it is something someone assigns.

Proposition is a property (mathematical result) that one can derive easily or directly from a given definition of an object.

Lemma is a mathematical result that is useful in establishing the truth of some other result.It is usually technical in nature and is not of primary importance to the overall body of knowledge one is trying to develop.

Theorem is a true mathematical statement whose truth can be verified. It is a property of major importance that one can derive which usually has far-sweeping consequences for the area of math one is studying. Theorems don't necessarily need the support of propositions or lemmas, but they often do require other smaller results to support their evidence.

Corollary is a mathematical result that can be deduced from, and is thereby a consequence of, some earlier result. It is usually a direct consequence of a major theorem.

Conjecture is an educated prediction that one makes based on their experience.

Integers and some of their basic properties and definitions

Let $a, b, c \in \mathbb{Z}$:

Axiom	w.r.t.addition	w.r.t. multiplication
Closure	$a + b \in \mathbb{Z}$	$a \cdot b \in \mathbb{Z}$
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Commutative	$a + b = b + a$	$ab = ba$
Distributive	$a(b + c) = ab + ac$	
Identity element	$a + 0 = a$	$a \cdot 1 = a$ Note: $0 \neq 1$ and $a \cdot 0 = 0$.
Inverse element	There exists a unique integer $-a = (-1) \cdot a$ such that $a + (-a) = 0$	

property	w.r.t.addition	w.r.t. multiplication
Subtraction	$b - a := b + (-a)$	
No divisors of 0 **(The zero product property)**		If $ab = 0$ then $a = 0$ or $b = 0$.
Cancellation Law	If $a + c = b + c$, then $a = b$.	If $ab = ac$ and $a \neq 0$, then $b = c$.

Order properties:

1. If $a < b$ and $b < c$ then $a < c$. (**transitivity**)
2. Exactly one of $a < b$ or $a = b$ or $a > b$ holds. (**trichotomy**)
3. If $a < b$, then $a + c < b + c$.
4. If $c > 0$, then $a < b$ iff $ac < bc$. If $c < 0$, then $a < b$ iff $ac > bc$.

Mathematical definitions are always <u>biconditional</u> statements.

Recall that by Corollary 20

$$P \Rightarrow Q \equiv (\neg Q) \Leftrightarrow (\neg P)$$

DEFINITION A. *An integer n is defined to be* **even** *if $n = 2k$ for some integer k. An integer n is defined to be* **odd** *if $n = 2k + 1$ for some integer k.*

EXAMPLE 37. *Fill in the blanks:*

- *The number 8 is even because $8 = 2(\_\_\_)$. Here, ___ plays the role of n and ___ plays the role of k in the definition.*
- *The number -11 is even because $-11 = 2(\_\_\_) + 1$. Here, ___ plays the role of n and ___ plays the role of k in the definition.*

DEFINITION B. The **parity** of an integer refers to its oddness or evenness. *Two integers are said to be* **of the same parity** *if they are both even, or both odd. Two integers are said to be* **of opposite parity** *if one of them is even and the other is odd.*

FACT *Every integer is either even, or odd.*

DEFINITION C. For any integer n, we call the pair of integers n and $n + 1$ consecutive.

DEFINITION D. Let a and b be integers. We say that b **divides** a, written $b|a$, if there is an integer c such that $bc = a$. We say that b and c are **factors** of a, or that a is **divisible** by b and c.

DIRECT PROOFS

To prove (directly) a universal statement "For all $x \in D$, $S(x)$ is true":

- Assume x is an arbitrary (but now fixed) element $x \in D$.
- Demonstrate that $S(x)$ is true.

EXAMPLE 38. *Let $n \in \mathbb{E}$. Prove that $5n^5 + n + 6$ is even.*

Proof. Let $n \in \mathbb{E}$. Since n is even, there is an integer k for which _____. Now we get

$$5n^5 + n + 6 = \underline{\hspace{6cm}} = \underline{\hspace{5cm}}.$$

Therefore $5n^5 + n + 6$ is even, because ______________________. □

To prove (directly) a universal conditional statement "For all $x \in D$, $P(x) \Rightarrow Q(x)$":

- Assume that $P(x)$ is true for an arbitrary (but now fixed) element $x \in D$.
- Draw out consequences of $P(x)$.
- Use these consequences to show that $Q(x)$ must be true as well for this element x.

REMARK 39. Note that if $P(x)$ is false for some $x \in D$, then $P(x) \Rightarrow Q(x)$ is ________ for this element x. This is why we need only be concerned with showing that $P(x) \Rightarrow Q(x)$ is true for all $x \in D$ for which $P(x)$ is true.

EXAMPLE 40. The following is an attempted proof of a result. What is the result and is the attempted proof correct?

Proof. Let a be an even integer and b be an odd integer. Then $a = 2n$ and $b = 2n+1$ for some integer n. Therefore,

$$3a - 5b = 3(2n) - 5(2n+1) = 6n - 10n - 5 = -4n - 5 = 2(-2n-2) - 1.$$

Since $-2n - 2$ is an integer, $3a - 5b$ is odd. □

THEOREM 41. *1. The sum and product of every two even integers is even.*

2. The sum of every two odd integers is even.

3. The product of every two odd integers is odd.

HINT: First express the statements in the form *"For all ..., if ... then..."* using symbols to represent variables.

EXAMPLE 42. *Let $a, b, c, d \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$. Prove the following:*

(a) *If $a|b$ and $b|c$, then $a|c$.*

(b) *If $a|c$ and $b|d$, then $ab|cd$.*

PROOF BY CASES

may be useful while attempting to give a proof of a statement concerning an element x in some set D. Namely, if x possesses one of two or more properties, then it may be convenient to divide a case into other cases, called *subcases*.

Result	Possible cases
$\forall n \in \mathbb{Z}, R(n)$	`Case 1.` $n \in \mathbb{E}$; `Case 2.` ______________
$\forall x \in \mathbb{R}, Q(x)$	`Case 1.` $x < 0$; `Case 2.` ______________ `Case 3.` $x > 0$
$\forall n \in \mathbb{Z}^+, P(n)$	`Case 1.` ______________; `Case 2.` $n \geq 2$.

EXAMPLE 43. *Prove that the product of any two consecutive integers is even.*

Disproving Statements

Case 1. Counterexamples

Let $S(x)$ be a predicate over a domain D. If the quantified statement $(\forall x \in D, S(x).)$ is *false*, then its negation is true, i.e.

Such an element x is called a **counterexample** of the false statement $\forall x \in D, S(x)$.

EXAMPLE 44. ***Disprove** the statement:* *"If $n \in \mathbb{O}$, then $3|n^2+2$."*

Solution.

EXAMPLE 45. *Negate the statement: "For all $x \in D, P(x) \Rightarrow Q(x)$."*

The value assigned to the variable x that makes $P(x)$ true and $Q(x)$ false is a **counterexample** of the statement "For all $x \in D, P(x) \Rightarrow Q(x)$."

EXAMPLE 46. *S: If n is an integer and n^2 is a multiple of 4 then n is a multiple of 4.*
Question: Is the following "proof" valid?
Let $n = 6$. Then $n^2 = 6^2 = 36$ and 36 is a multiple of 4, but 6 is not a multiple of 4. Therefore, the statement S is FALSE.□

EXAMPLE 47. Disprove the following statement:

If a real-valued function is continuous at some point, then this function is differentiable there.

Case 2: Existential Statements

Consider the quantified statement $\exists x \in D \ni S(x)$. If this statement is *false*, then its negation is true, i.e.

EXAMPLE 48. *Disprove the statement:* "There exist an even integer n such that $3n + 5$ is even."

Summary

- To disprove a universal statement, provide a counterexample.
- To disprove an existential statement, formulate a negation of it and provide a proof of the negation.

2.2 Indirect proofs: Proofs by contradiction and contrapositive

Contrapositive

Recall that the statement $\neg Q \Rightarrow \neg P$ is called the **contrapositive** of the statement $P \Rightarrow Q$. Moreover,

$$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P.$$

In other words, in order to prove $P \Rightarrow Q$, we may choose instead to prove $\neg Q \Rightarrow \neg P$.

PROOF BY CONTRAPOSITIVE

Let $P(x)$ and $Q(x)$ be predicates over a domain D. **To prove a universal conditional statement "for all $x \in D$, $P(x) \Rightarrow Q(x)$"**

- Assume that $\neg Q(x)$ is true for an arbitrary (but now fixed) element $x \in D$.
- Draw out consequences of $\neg Q(x)$.
- Use these consequences to show that $\neg P(x)$ must be true as well for this element x.
- It follows that $P(x) \Rightarrow Q(x)$ for all $x \in D$.

REMARK 49. If you use a contrapositive method, you must declare it in the beginning and then state **what is sufficient to prove.**

EXAMPLE 50. *Let x be an integer. If $5x - 7$ is even, then x is odd.*

EXAMPLE 51. *Let $x, y \in \mathbb{Z}$. If $7 \nmid xy$, then $7 \nmid x$ and $7 \nmid y$.*

Proving biconditional statements

Prove that $\forall x \in D, P(x) \Leftrightarrow Q(x)$.

Proof. Let $x \in D$.

Assume $P(x)$. Then show $Q(x)$.

Conversely, assume $Q(x)$. Then show $P(x)$.□

EXAMPLE 52. *Let $x, y \in \mathbb{Z}$. Prove that x and y are of opposite parity if and only if $x + y$ is odd.*

THEOREM 53. *Let n be an integer. Then n is even if and only if n^2 is even.*

Proof.

COROLLARY 54. *Let n be an integer. Then n is odd iff n^2 is odd.*

COROLLARY 55. *For every integer n, both n and n^2 are of the same parity.*

EXAMPLE 56. *Let $x \in \mathbb{Z}$. Prove that if $2|(x^2-1)$ then $4|(x^2-1)$.*

PROOF BY CONTRADICTION

To prove a statement S is true by contradiction:

- Assume that $\neg S$ is true.
- Deduce a contradiction.
- Then conclude that S is true.

REMARK 57. If you use a proof by contradiction to prove that S, you should alert the reader about that by saying (or writing) one of the following

- Suppose that the statement S is false.
- Assume, to the contrary, that the statement S is false.
- By contradiction, assume, that the statement S is false.

THEOREM 58. *Every integer is either even, or odd.*

REMARK 59. If you use a proof by contradiction to prove a universal conditional statement

$$\forall x \in D, P(x) \Rightarrow Q(x),$$

then the proof begins by assuming the existence of a counterexample of this statement. Therefore, the proof might begin with one of the following.

- Assume, to the contrary, that there exists some element $x \in D$ for which $P(x)$ is true and $Q(x)$ is false.
- By contradiction, assume, that there exists an element $x \in D$ such that $P(x)$ is true, but $\neg Q(x)$ is true.

PROPOSITION 60. *If m and n are integers, then $m^2 \neq 4n + 2$.*

COROLLARY 61. *The equation $m^2 - 4n = 2$ has no integer solutions.*

COROLLARY 62. *If the square of an integer is divided by 4, the remainder cannot be equal 2.*

COROLLARY 63. *The square of an integer cannot be of the form $4n + 2$, $n \in \mathbb{Z}$.*

Proof of the Proposition 60.

Summary of Three Proof Techniques

How to prove that $\forall x \in D, P(x) \Rightarrow Q(x)$.

Technique	direct proof	proof by contrapositive	proof by contradiction
Assume			
Goal			

Existence Proofs

An existence theorem can be expressed as a quantified statement
$\exists x \in D \ni S(x)$:

$$\text{There exists } x \in D \text{ such that } S(x) \text{ is true.}$$

A proof of an existence theorem is called an existence proof.

One simple way to prove existence is to provide an object that has the desired property This sort of proof is called **constructive** proof (see Example 64 below). But not all existence proofs are constructive can prove existence through other methods (e.g., proof by contradiction) Such indirect existence proofs called nonconstructive proofs (see Example 65 below).

EXAMPLE 64. *There exist real numbers a and b such that $\sqrt{a^2+b^2} = a+b$.*

Proof.

Recall that the *Intermediate Value Theorem of Calculus* implies that if a continuous function takes values of opposite sign inside an interval, then it has a root in that interval.

EXAMPLE 65. *Prove that the polynomial $f(x) = x^3 + x^2 - 1$ has a real root between $x = 2/3$ and $x = 1$.*

Proof.

Uniqueness Proof

An element belonging to some prescribed set D and possessing a certain property P is **unique** if it is the only element of D having property P. A typical way to prove uniqueness is a proof by contradiction: *Assume that x and y are distinct elements of D and show that $x = y$.*

EXAMPLE 66. *If a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.*

3.1 Principle of Mathematical Induction

"Domino Effect"

Step 1. The first domino falls.

Step 2. When any domino falls, the next domino falls.

Conclusion. All dominoes will fall!

THEOREM 67. *(*`Principle of Mathematical Induction (PMI)`*) Let $P(n)$ be a statement about the positive integer n so that n is a free variable in $P(n)$.* ***Suppose the following:***

- *The statement $P(1)$ is true.*

- *For all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.*

Then*, for all positive integers n, $P(n)$ is true.*

Strategy

The proof by induction consists of the following steps:

Base Case: Verify that $P(1)$ is true.

Inductive hypothesis: Assume that k is a positive integer for which $P(k)$ is true .

Inductive Step: With the assumption made, prove that $P(k+1)$ is true.

Conclusion: $P(n)$ is true for every positive integer n.

EXAMPLE 68. *Prove by induction the formula for the sum of the first n positive integers*

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}. \tag{5}$$

Proof. Given a positive integer n, let $P(n)$ be the statement

$$1 + 2 + \ldots n = \frac{n(n+1)}{2}.$$

Base Case: Since $1 = \frac{1(1+1)}{2}$, we conclude $P(1)$ is true.

Induction hypothesis: Assume that k is a natural number for which $P(k)$ is true, i.e.

(6)

Induction step: With the assumption made, establish that $P(k+1)$ is true, which is equivalent to showing that

(7)

To show that $P(k+1)$ is true, notice that $1+2+\ldots k+(k+1) =$

Thus $P(k+1)$ is true.

Conclusion: It now follows by Principle of Mathematical Induction that $P(n)$ is true for every positive integer n.$\square$

EXAMPLE 69. *Prove that* $3|(8^n - 5^n)$ *for every positive integer* n.

EXAMPLE 70. *Find the sum of all odd numbers from* 1 *to* $2n+1$ $(n \in \mathbb{Z}^+)$.

THEOREM 71. *(`The Strong Principle of Mathematical Induction`) Let* $P(n)$ *be a statement about the positive integer* n. *Suppose that* $P(1)$ *is true. Whenever* k *is a positive integer for which* $P(i)$ *is true for every positive integer* i *such that* $i \leq k$, *then* $P(k+1)$ *is true. Then* $P(n)$ *is true for every positive integer* n.

Strategy

The proof by the Strong Principle of Mathematical Induction consists of the following steps:

Base Case: Verify that $P(1)$ is true.

Induction hypothesis: Assume that k is a positive integer for which $P(1), P(2), \ldots, P(k)$ are true .

Inductive Step: With the assumption made, prove that $P(k+1)$ is true.

Conclusion: $P(n)$ is true for every positive integer n.

EXAMPLE 72. *Let* $a_1, a_2, a_3, a_4, \ldots$ *be a recursively defined infinite sequence. Namely,* $a_1 = 1, a_2 = 2, a_3 = 3$*, and* $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ *for all* $n \geq 4$*. Prove that* $a_n < 2^n$ *for all positive integers* n*.*