Math 662 Homework Assignment 2

Due Friday October 11

1. (An algebraic Mayer-Vietoris sequence.) Let R be a ring, let A, B be submodules of a left R-module M, and let C be another left R-module. Prove that there is a long exact sequence

$$0 \to \mathrm{Hom}_R(A+B, C) \to \mathrm{Hom}_R(A, C) \oplus \mathrm{Hom}_R(B, C) \to \mathrm{Hom}_R(A \cap B, C) \to$$
$$\mathrm{Ext}^1_R(A+B, C) \to \mathrm{Ext}^1_R(A, C) \oplus \mathrm{Ext}^1_R(B, C) \to \mathrm{Ext}^1(A \cap B, C) \to \cdots$$

2. (a) Let A be a right R-module and B a left R-module. If either $\mathrm{pd}(A) = m$ or $\mathrm{pd}(B) = m$, show that $\mathrm{Tor}^R_n(A, B) = 0$ for all $n \geq m + 1$.
   Define the *Tor dimension* of R to be

   $$\mathrm{Tor}\dim(R) = \sup\{d \mid \mathrm{Tor}^R_d(A, B) \neq 0 \text{ for some } R\text{-modules } A \text{ and } B\}.$$

   (b) Prove that $\mathrm{Tor}\dim(R) \leq \mathrm{gldim}(R)$. (There are non-noetherian rings R for which this is a strict inequality.)

3. Let R be a ring, and let A, B be R-modules. Prove that

$$\mathrm{pd}_R(A \oplus B) = \max\{\mathrm{pd}_R(A),\ \mathrm{pd}_R(B)\}.$$

4. Let $S \subseteq R$ be rings, B an S-module and A an R-module. Let A also denote restriction of A to an S-module, and $R \otimes_S B$ induction of B to an R-module.
   (a) Prove that $\mathrm{Hom}_S(B, A) \cong \mathrm{Hom}_R(R \otimes_S B, A)$.
   (b) Prove the Eckmann-Shapiro Lemma: If R is a projective right S-module, under multiplication, then $\mathrm{Ext}^n_S(B, A) \cong \mathrm{Ext}^n_R(R \otimes_S B, A)$.