Math 662 Homework Assignment 3

Due Friday November 1

1. Let B be an R-module for a ring R. Prove that the functor $-\otimes_R B$, from Mod-R to $\mathbb{Z}$-Mod, is additive.

2. Let R and S be rings that are Morita equivalent via an additive functor F : Mod-$R \to$ Mod-S. Prove that $\mathrm{gldim}(R) = \mathrm{gldim}(S)$ and $\mathrm{Ext}^n_R(A, B) \cong \mathrm{Ext}^n_S(F(A), F(B))$ as abelian groups for all R-modules A, B and all $n \geq 0$.

3. Let $R = k[x, y]$ where k is a field. Note that $k \cong R/(x, y)$ as rings and consider k to be an R-module via the quotient map (i.e. x, y each act as 0 on elements of k). Find $\mathrm{Ext}^n_R(k, k)$ for all $n \geq 0$ (that is, describe it as a vector space) and $\mathrm{pd}_R(k)$. (Hint: Recall from class the following free resolution of k as an R-module:
$$0 \to R \xrightarrow{\alpha} R \oplus R \xrightarrow{\beta} R \to k \to 0$$
where $\alpha = \begin{pmatrix} y \\ -x \end{pmatrix}$ and $\beta = (x \;\; y)$.)

4. Let k be a field and let $R = k[x, y]/(x^2, y^2)$. Consider k to be an R-module on which the images of x and y each act as multiplication by 0.
   (a) Find a free R-module resolution of k. (Hint: Recall the free $k[x]/(x^2)$-resolution of k from class. Take the tensor product, over k, of this complex with a similar complex for $k[y]/(y^2)$. Apply the Künneth Theorem to see that its total complex is exact in all degrees but 0. Noting that all maps are in fact R-module homomorphisms, conclude that this is a free R-resolution of k.)
   (b) Note that the modules in the resolution in part (a) are k-vector spaces. What is the dimension of the vector space in degree n?