Math 220 Practice for Exam 1

1. Consider the statement: For all integers m and n, if m is even and n is even, then $m+n$ is divisible by 4. False: Counterexample: $m=2$, $n=4$, $m+n=6$ $\quad P \Rightarrow Q$

(a) Write the converse of this statement.

For all integers m and n,

if $m+n$ is divisible by 4, then m is even and n is even. $\quad Q \Rightarrow P$

False: Counterexample $m=7$, $n=9$, $m+n=16$

(b) Write the contrapositive of this statement.

For all integers m and n, $\quad (\neg Q) \Rightarrow (\neg P)$

if $m+n$ is not divisible by 4, then

m is odd or n is odd.

False: Counterexample: $m=2, n=4$, $m+n=6$

(c) Write the negation of this statement.

There exist integers m and n such that $\quad P \wedge (\neg Q)$

m is even and n is even

and $m+n$ is not divisible by 4.

True: Proof $m=2$, $n=4$, $m+n=6$.

(d) [5] Which of the above four statements *(the proposition, its converse (a), its contrapositive (b), its negation (c))* are true? (You need not justify your answer.)

negation (c)

2. Consider the statement: For all real numbers x and y, if xy is rational, then x is rational. False. Counterexample: $x=\sqrt{2}$, $y=\sqrt{2}$, $xy=2$
irrational rational

(a) Write the converse of this statement.

For all real numbers x and y,
if x is rational, then xy is rational.

False. Counterexample: $x=1$, $y=\sqrt{2}$, $xy=\sqrt{2}$

(b) Write the contrapositive of this statement.

For all real numbers x and y,
if x is irrational, then xy is irrational.

False. Counterexample: $x=\sqrt{2}$, $y=\sqrt{2}$, $xy=2$

(c) Write the negation of this statement.

There exist real numbers x and y such that
xy is rational and x is irrational.

True: $x=\sqrt{2}$, $y=\sqrt{2}$, $xy=2$

(d) Which of the above four statements *(the proposition, its converse (a), its contrapositive (b), its negation (c))* are true? (You need not justify your answer.)

negation (c)

3. Prove that for all integers m and n, if m and n are both odd, then $m+n$ is even. Is the converse true?

Proof: Let m and n be integers.

Assume m and n are both odd, i.e. $m = 2k+1$ and $n = 2j+1$ for some $j, k \in \mathbb{Z}$.

Then
$$\begin{aligned} m+n &= (2k+1) + (2j+1) \\ &= 2k + 2j + 2 \\ &= 2(k+j+1), \end{aligned}$$
which is even since $k+j+1$ is an integer. $\square$

Converse: For all integers m and n,

if $m+n$ is even, then m and n are both odd.

False: Counterexample

$m = 2$, $n = 2$, $m+n = 4$

4. Prove that for all integers n, n is divisible by 3 if, and only if, n^2 is divisible by 3.

biconditional statement

Proof ($\Rightarrow$)

We'll prove that for all integers n, if n is divisible by 3, then n^2 is divisible by 3.

Proof: Let n be an integer.

Assume n is divisible 3, i.e. $n = 3k$ for some integer k.

So $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$,

which is divisible by 3 since $3k^2$ is an integer. $\square$

($\Leftarrow$)

We'll prove that for all integers n, if n^2 is divisible by 3, then n is divisible by 3.

Proof: HW - did in class.

5. Prove there do not exist integers m and n for which $9m + 27n = 2$.

Proof (by contradiction):

Assume there exist integers m and n for which $9m+27n=2$.

So

$$9(m+3n) = 2$$

The left side expression is divisible by 9 (and so also divisible by 3), but the right side expression is not. This is a contradiction, and therefore there do not exist such integers m and n. $\square$

6. (a) [5] State the definition of *limit*, that is $\lim_{x \to a} f(x) = L$ means

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all real numbers x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

(b) [15] Prove $\lim_{x \to 3} (1 - 4x) = -11$ using the definition of limit. $(a = 3, \quad L = -11)$

Proof: Let $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{4}$. ← divide ε by absolute value of slope

Let x be a real number.

Assume

$0 < |x - 3| < \delta$.

Then

$$
\begin{aligned}
|(1-4x) - (-11)| &= |1 - 4x + 11| \\
&= |12 - 4x| \\
&= 4|3 - x| \\
&= 4|x - 3| \\
&< 4\delta \\
&= 4 \cdot \frac{\varepsilon}{4} \\
&= \varepsilon .
\end{aligned}
$$

$$\left(\begin{aligned} |3 - x| &= |(-1)(x-3)| \\ &= |x - 3| \end{aligned} \right)$$

Therefore $\lim_{x \to 3} (1 - 4x) = -11$. $\square$