Some homework solutions

1.2 #1a Let n be an even integer, so that $n = 2k$ for some integer k. Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even since $2k^2$ is an integer.

#1b Let n be an odd integer, so that $n = 2k+1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd since $2k^2 + 2k$ is an integer.

#2a Let m and n be even integers, so that $m = 2k$ and $n = 2l$ for some integers k and l. Then

$$m + n = 2k + 2l = 2(k+l),$$

which is even since $k + l$ is even.

#2b Let m and n be odd integers, so that $m = 2k+1$ and $n = 2l+1$ for some integers k and l. Then

$$m+n = (2k+1) + (2l+1) = 2k + 2l + 2 = 2(k+l+1),$$

which is even since $k + l + 1$ is an integer.

#3a Let m and n be integers. Assume m is even, so that $m = 2k$ for some integer k. Then $mn = (2k)n = 2(kn)$, which is even since kn is an integer.

#3b Let m and n be odd integers, so that $m = 2k+1$ and $n = 2l+1$ for some integers k and l. Then

$$mn = (2k+1)(2l+1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1,$$

which is odd since $2kl + k + l$ is an integer.

#4 Let n be an integer.

Case 1 n is even, so that $n = 2k$ for some integer k.

Then $4n + 7 = 4(2k) + 7 = 8k + 7 = 8k + 6 + 1 = 2(4k+3) + 1$, which is odd since $4k+3$ is an integer.

Case 2 n is odd, so that $n = 2k+1$ for some integer k.

Then
$$\begin{aligned} 4n + 7 &= 4(2k+1) + 7 \\ &= 8k + 4 + 7 \\ &= 8k + 11 \\ &= 8k + 10 + 1 \\ &= 2(4k+5) + 1, \end{aligned}$$

which is odd since $4k + 5$ is an integer.

2.1 #1 Let a, b, c, m, n be integers. Assume that $a|b$ and $a|c$, so that $b = ak$ and $c = al$ for some integers k and l. Then
$$bm + cn = akm + aln = a(km + ln),$$
so $a|(bm+cn)$.

2.1 #3 Let m be an odd integer, that is, $m = 2n+1$ for some integer n.
Then $m^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 4(n^2+n)+1$.
Case 1 n is even, i.e. $n = 2a$ for some integer a.
Then $m^2 = 4(4a^2+2a)+1 = 8(2a^2+a)+1$.
Let $k = 2a^2+a$, and then $m^2 = 2k+1$ where k is an integer.
Case 2 n is odd, i.e. $n = 2b+1$ for some integer b.
Then
$$\begin{aligned} m^2 &= 4((2b+1)^2 + (2b+1)) + 1 \\ &= 4(4b^2+4b+1+2b+1) + 1 \\ &= 8(2b^2+3b+1) + 1. \end{aligned}$$
Let $k = 2b^2+3b+1$, and then $m^2 = 2k+1$ where k is an integer.

2.1 #4 Let n be an integer.
Case 1 n is even, i.e. $n = 2k$ for some integer k. Then
$$n^2+n+5 = (2k)^2 + (2k) + 5 = 4k^2 + 2k + 4 + 1 = 2(2k^2+k+2)+1,$$
which is odd since $2k^2+k+2$ is an integer.
Case 2 n is odd, i.e. $n = 2k+1$ for some integer k. Then
$$\begin{aligned} n^2+n+5 &= (2k+1)^2 + (2k+1) + 5 \\ &= 4k^2 + 4k + 1 + 2k + 6 \\ &= 2(2k^2+3k+3) + 1, \end{aligned}$$
which is odd since $2k^2+3k+3$ is an integer.

2.1 #5e False. Counterexample: $a=8$, $b=2$, $c=4$
#5f False. Counterexample: $a=4$, $b=6$, $c=2$
#5g True. Proof: Let m and n be even integers, i.e. $m=2k$ and $n=2j$ for some integers k, j. Then $mn = (2k)(2j) = 4kj$, which is divisible by 4.

2.1 #8 Let x and y be real numbers. Then
$$\begin{aligned} x^2+xy+y^2 &= x^2 + xy + \tfrac{1}{4}y^2 - \tfrac{1}{4}y^2 + y^2 \\ &= \left(x + \tfrac{1}{2}y\right)^2 + \tfrac{3}{4}y^2 \end{aligned}$$
Since x, y are real numbers, so is $x + \frac{1}{2}y$, and so $(x+\frac{1}{2}y)^2 \geq 0$. Since y is a real number, $y^2 \geq 0$, and so $\frac{3}{4}y^2 \geq 0$. Adding these two expressions, we obtain
$$\left(x+\tfrac{1}{2}y\right)^2 + \tfrac{3}{4}y^2 \geq 0,$$ as desired.