Math 220 Exam 2 Practice Problems
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The following are some representative problems, from old exams, on the material for Exam 2. They are not meant to include examples of all possible problems that may be on the exam. You will also want to be prepared to work any problems similar to homework problems or those from class.

1. Prove by induction that for each positive integer n,

$$P(n): \quad 2 + 6 + 10 + \cdots + (4n+2) = 2(n+1)^2.$$

Proof: 1. The statement $P(1)$ is true: $2+6 = 8$ and $2(1+1)^2 = 2\cdot 2^2 = 2\cdot 4 = 8$

2. Assume that for some positive integer m, $P(m)$ is true, that is
$$2+6+10+\cdots+(4m+2) = 2(m+1)^2.$$

Then

$$\begin{aligned}
&2+6+10+\cdots+(4m+2)+(4(m+1)+2) \\
&= 2(m+1)^2 + (4(m+1)+2) \\
&= 2(m^2+2m+1) + (4m+6) \\
&= 2m^2+4m+2+4m+6 \\
&= 2m^2+8m+8 \\
&= 2(m^2+4m+4) \\
&= 2(m+2)^2 \\
&= 2((m+1)+1)^2,
\end{aligned}$$

So $P(m+1)$ is true. By math induction, $P(n)$ is true for all positive integers n. $\square$

2. Prove by induction that for all integers $n \geq 1$, $\underbrace{\text{3 divides } n^3 + 2n}_{P(n)}$.

Proof: 1. If $n=1$, $n^3+2n = 1^3+2\cdot 1 = 3$, which is divisible by 3, so $P(1)$ is true.

2. Assume that for some integer $m \geq 1$, 3 divides m^3+2m. Then

$$(m+1)^3 + 2(m+1) = m^3+3m^2+3m+1 + 2m+2$$
$$= (m^3+2m) + (3m^2+3m+3)$$
$$= (m^3+2m) + 3(m^2+m+1).$$

By the induction hypothesis, m^3+2m is divisible by 3. The term $3(m^2+m+1)$ is divisible by 3 since m^2+m+1 is an integer. Therefore the sum $(m^3+2m) + 3(m^2+m+1)$ is divisible by 3, that is, $(m+1)^3+2(m+1)$ is divisible by 3. By induction, 3 divides n^3+2n for all integers $n \geq 1$. $\square$

3. Prove by induction that for each natural number n,

$$P(n): \quad 1+4+7+\ldots+(3n+1)=\frac{(n+1)(3n+2)}{2}.$$

Proof 1. If $n=1$, the left side of the equation is $1+4=5$, and the right side of the equation is $\frac{(1+1)(3\cdot 1+2)}{2}=\frac{2\cdot 5}{2}=5.$

So $P(1)$ is true.

2. Assume that for some natural number m, $P(m)$ is true, i.e.

$$1+4+7+\ldots+(3m+1)=\frac{(m+1)(3m+2)}{2}.$$

Then

$$\begin{aligned}
&1+4+7+\ldots+(3m+1)+(3(m+1)+1)\\
&=\frac{(m+1)(3m+2)}{2}+(3m+4)\\
&=\frac{3m^2+5m+2+6m+8}{2}\\
&=\frac{3m^2+11m+10}{2}\\
&=\frac{(m+2)(3m+5)}{2}\\
&=\frac{((m+1)+1)(3(m+1)+2)}{2},
\end{aligned}$$

that is, $P(m+1)$ is true. By induction, $P(n)$ is true for all natural numbers n. $\square$

4. Let $a_1 = 2$, $a_2 = 1$, and $a_n = 3a_{n-1} + a_{n-2}$ for each natural number $n \geq 3$. Prove by induction that for each natural number n [redacted], $\underbrace{a_n \geq 3^{n-2}}_{P(n)}$.

Proof 1. $a_1 = 2$, which is greater than $3^{1-2} = \frac{1}{3}$, so $P(1)$ is true.
$a_2 = 1$, which is greater than $3^{2-2} = 3^0 = 1$, so $P(2)$ is true.

2. Assume that for some natural number $m \geqslant 2$,
$$a_k \geqslant 3^{k-2} \quad \text{for all } k \text{ for which } 1 \leq k \leq m.$$

Then
$$\begin{aligned} a_{m+1} &= 3a_m + a_{m-1} \\ &\geqslant 3 \cdot 3^{m-2} + 3^{m-1-2} \quad \text{(by the induction hypothesis)} \\ &> 3^{m-1} \quad (\text{since } 3^{m-1-2} > 0) \\ &= 3^{(m+1)-2}, \end{aligned}$$
that is, $P(m+1)$ is true. By strong mathematical induction, $P(n)$ is true for all natural numbers n.

5. Let $a_1 = 1$, $a_2 = 9$, and $a_{n+1} = 9a_n - 20a_{n-1}$ for all $n \geq 2$. Prove that for all positive integers n, $\underbrace{a_n = 5^n - 4^n}_{P(n)}$.

Proof 1. $a_1 = 1 = 5^1 - 4^1$, so $P(1)$ is true.

$a_2 = 9 = 5^2 - 4^2$, so $P(2)$ is true.

2. Assume that for some positive integer m, $a_k = 5^k - 4^k$ for all positive integers k for which $1 \leq k \leq m$. Then

$$
\begin{aligned}
a_{m+1} &= 9a_m - 20a_{m-1} \\
&= 9(5^m - 4^m) - 20(5^{m-1} - 4^{m-1}) \\
&= 45 \cdot 5^{m-1} - 36 \cdot 4^{m-1} - 20 \cdot 5^{m-1} + 20 \cdot 4^{m-1} \\
&= 25 \cdot 5^{m-1} - 16 \cdot 4^{m-1} \\
&= 5^{m+1} - 4^{m+1},
\end{aligned}
$$

that is, $P(m+1)$ is true. By strong mathematical induction, $P(n)$ is true for all positive integers n. $\square$

6. Let $a_1 = 3$, $a_2 = 5$, and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for all $n \geq 2$. Prove that for all positive integers n, $\underbrace{3 \leq a_n \leq 5}_{P(n)}$.

Proof 1. $a_1 = 3$, so $3 \leq a_1 \leq 5$.

$a_2 = 5$, so $3 \leq a_2 \leq 5$.

We have thus shown that $P(1)$ and $P(2)$ are true.

2. Assume that for some integer $m \geq 2$,

$$3 \leq a_k \leq 5$$

for all integers k for which $1 \leq k \leq m$.

Then

$$a_{m+1} = \frac{1}{2}(a_m + a_{m-1}),$$

and since $3 \leq a_m \leq 5$ and $3 \leq a_{m-1} \leq 5$, it follows that

$$\frac{1}{2}(3+3) \leq \frac{1}{2}(a_m + a_{m-1}) \leq \frac{1}{2}(5+5)$$

$$\frac{6}{2} \leq \frac{1}{2}(a_m + a_{m-1}) \leq \frac{10}{2}$$

$$3 \leq \frac{1}{2}(a_m + a_{m-1}) \leq 5$$

That is, $3 \leq a_{m+1} \leq 5$, so $P(m+1)$ is true. By strong math induction, $P(n)$ is true for all positive integers n. □

7. Consider the following two sets:

$$\begin{aligned} S &= \{n \in \mathbb{Z} \mid n = 3x + 6y \text{ for some } x, y \in \mathbb{Z}\}, \\ T &= \{n \in \mathbb{Z} \mid n = 3x + 2y \text{ for some } x, y \in \mathbb{Z}\} \end{aligned}$$

(a) Is $S \subseteq T$? Justify your answer.

Rewrite $T = \{n \in \mathbb{Z} \mid n = 3r + 2s \text{ for some } r, s \in \mathbb{Z}\}$.

Yes: let $n \in S$, that is, $n = 3x + 6y$ for some $x, y \in \mathbb{Z}$. Then

$$n = 3x + 2(3y).$$

Set $r = x$ and $s = 3y$, so that $n = 3r + 2s$. It follows that $n \in T$ since r and s are integers. Therefore $S \subseteq T$.

(b) Is $T \subseteq S$? Justify your answer.

No: For example, $5 \in T$ since $5 = 3 \cdot 1 + 2 \cdot 1$. However, $5 \notin S$. If it were, that is, if $5 = 3x + 6y$ for some $x, y \in \mathbb{Z}$, then $5 = 3(x + 2y)$. This is a contradiction since $3(x+2y)$ is divisible by 3, and 5 is not divisible by 3. $\square$

8. Consider the following statement.

P: For all sets A and B, $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

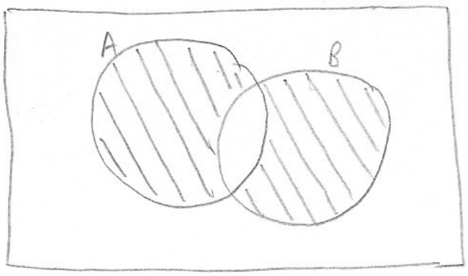
(a) *Just for this part*, let $A = \{1, 2, 3, 4\}$ and $B = \{0, 2, 4\}$. Find the following sets:

$A \cup B = \{0, 1, 2, 3, 4\}$ $\qquad$ $A \cap B = \{2, 4\}$.

$A - B = \{1, 3\}$ $\qquad$ $B - A = \{0\}$

$(A \cup B) - (A \cap B) = \{0, 1, 3\}$ $\qquad$ $(A - B) \cup (B - A) = \{0, 1, 3\}$

(b) Draw a Venn diagram to illustrate the statement P in general.



(c) Prove the statement P.

Proof that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$:

Let $x \in (A \cup B) - (A \cap B)$, that is, $x \in A \cup B$ and $x \notin A \cap B$.

Then $x \in A$ or $x \in B$ and $x \notin A \cap B$.

Case 1 $x \in A$. Then, since $x \notin A \cap B$, we conclude that $x \notin B$. Therefore $x \in A - B$.

Case 2 $x \in B$. Then, since $x \notin A \cap B$, we conclude that $x \notin A$. So $x \in B - A$.

In either case, $x \in (A - B) \cup (B - A)$. So $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$.

Proof that $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$:

Let $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$.

Case 1 $x \in A - B$. Then $x \in A$ and $x \notin B$.

Since $x \in A$, it is true that $x \in A \cup B$.

Since $x \notin B$, it is true that $x \notin A \cap B$.

Therefore $x \in (A \cup B) - (A \cap B)$.

Case 2 $x \in B - A$. This is similar to case 1.

In either case, $x \in (A \cup B) - (A \cap B)$. So $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.

Since we have proven both containments, we can conclude that the two sets are equal, that is

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A). \quad \square$$

9. (a) [5] Let $B = \{2,3,5,8\}$ and $C = \{3,7\}$. Find $B \times C$ (that is, write out all the elements of this set).

$\{(2,3), (3,3), (5,3), (8,3), (2,7), (3,7), (5,7), (8,7)\}$

(b) Prove that *for all sets* A, B, *and* C, if $A \subseteq B$, then $A \times C \subseteq B \times C$.

Proof Let A, B, C be sets for which $A \subseteq B$.
Let $(a,c) \in A \times C$, that is, $a \in A$ and $c \in C$.
Since $A \subseteq B$, it follows that $a \in B$. Therefore $(a,c) \in B \times C$,
and so $A \times C \subseteq B \times C$. $\square$

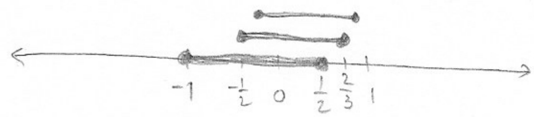
10. For each positive integer i, let $A_i = \left[-\frac{1}{i}, \frac{i}{i+1}\right]$. In the following, you need not prove that your answers are correct.

(a) Find $A_1 \cup A_2$ and $A_1 \cap A_2$.

$A_1 = [-1, \frac{1}{2}]$, $\quad A_2 = [-\frac{1}{2}, \frac{2}{3}]$

$A_1 \cup A_2 = [-1, \frac{2}{3}]$

$A_1 \cap A_2 = [-\frac{1}{2}, \frac{1}{2}]$



(b) Find $\bigcup_{i \in \mathbb{Z}^+} A_i$ and $\bigcap_{i \in \mathbb{Z}^+} A_i$.

As i gets larger, both left and right endpoints of A_i increase.

So $\bigcup_{i \in \mathbb{Z}^+} A_i = [-1, \lim_{i \to \infty} \frac{i}{i+1}] = [-1, 1]$

and $\bigcap_{i \in \mathbb{Z}^+} A_i = [\lim_{i \to \infty} (-\frac{1}{i}), \frac{1}{2}] = [0, \frac{1}{2}]$

11. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 1 - x^3$ for all $x \in \mathbb{R}$.
(a) Prove that f is bijective.

Injectivity: Let $x_1, x_2 \in \mathbb{R}$ for which $f(x_1) = f(x_2)$. Then $1 - x_1^3 = 1 - x_2^3$, so $x_1^3 = x_2^3$, which implies $x_1 = x_2$. Therefore f is injective.

Surjectivity: Let $y \in \mathbb{R}$. Set $x = \sqrt[3]{1-y}$. Then $f(x) = 1 - (\sqrt[3]{1-y})^3 = 1 - (1-y) = y$, so y is in the range of f. Therefore $\mathbb{R} \subseteq \operatorname{ran} f$. By definition of f (since $1 - x^3 \in \mathbb{R}$ for all $x \in \mathbb{R}$), $\operatorname{ran} f \subseteq \mathbb{R}$. Therefore $\operatorname{ran} f = \mathbb{R}$, that is, f is surjective.

(b) Find the inverse function f^{-1}.

$f^{-1}(x) = \sqrt[3]{1-x}$ (this is found by interchanging x and y in the equation $y = 1 - x^3$, and solving for y)

12. Let X, Y, Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Prove that if $g \circ f$ is injective, then f is injective.

Assume that $g \circ f$ is injective. Let $x_1, x_2 \in X$ for which $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is injective, this implies that $x_1 = x_2$. Therefore f is injective.

(b) Give an example of functions f and g for which $g \circ f$ is injective and g is not injective.

$f : [0, \infty) \to \mathbb{R}$ $\qquad$ $g : \mathbb{R} \to \mathbb{R}$

$f(x) = \sqrt{x}$ $\qquad$ $g(x) = x^2$

Then $(g \circ f)(x) = x$ for all $x \in [0, \infty)$, so $g \circ f$ is injective. However, g is not injective, since e.g. $g(-2) = g(2)$.

13. Let X, Y, Z be sets. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Prove that $g \circ f : X \to Z$ is invertible and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Let $x \in X$. Then

$$\begin{aligned}
(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) &= (f^{-1} \circ g^{-1})((g \circ f)(x)) \\
&= f^{-1}(g^{-1}(g(f(x)))) \\
&= f^{-1}(f(x)) && \text{since } g^{-1} \text{ is the inverse of } g \\
&= x && \text{since } f^{-1} \text{ is the inverse of } f.
\end{aligned}$$

Let $z \in Z$. Then

$$\begin{aligned}
(g \circ f) \circ (f^{-1} \circ g^{-1})(z) &= g(f(f^{-1}(g^{-1}(z)))) \\
&= g(g^{-1}(z)) && \text{since } f^{-1} \text{ is the inverse of } f \\
&= z && \text{since } g^{-1} \text{ is the inverse of } g.
\end{aligned}$$

Therefore, $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$, and $g \circ f$ is invertible.