Math 220 Exam 2 Practice Problems
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The following are some representative problems, from old exams, on the material for Exam 2. They are not meant to include examples of all possible problems that may be on the exam. You will also want to be prepared to work any problems similar to homework problems or those from class.

1. Prove by induction that for each positive integer n,

$$2 + 6 + 10 + \cdots + (4n + 2) = 2(n + 1)^2.$$

2. Prove by induction that for all integers $n \geq 1$, 3 divides $n^3 + 2n$.

3. Prove by induction that for each natural number n,

$$1 + 4 + 7 + \ldots + (3n + 1) = \frac{(n + 1)(3n + 2)}{2}.$$

4. Let $a_1 = 2$, $a_2 = 1$, and $a_n = 3a_{n-1} + a_{n-2}$ for each natural number $n \geq 3$. Prove by induction that for each natural number n, $a_n \geq 3^{n-2}$.

5. Let $a_1 = 1$, $a_2 = 9$, and $a_{n+1} = 9a_n - 20a_{n-1}$ for all $n \geq 2$. Prove that for all positive integers n, $a_n = 5^n - 4^n$.

6. Let $a_1 = 3$, $a_2 = 5$, and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for all $n \geq 2$. Prove that for all positive integers n, $3 \leq a_n \leq 5$.

7. Consider the following two sets:

$$\begin{aligned} S &= \{n \in \mathbb{Z} \mid n = 3x + 6y \text{ for some } x, y \in \mathbb{Z}\}, \\ T &= \{n \in \mathbb{Z} \mid n = 3x + 2y \text{ for some } x, y \in \mathbb{Z}\} \end{aligned}$$

(a) Is $S \subseteq T$? Justify your answer.

(b) Is $T \subseteq S$? Justify your answer.

8. Consider the following statement.

P: For all sets A and B, $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

(a) *Just for this part*, let $A = \{1, 2, 3, 4\}$ and $B = \{0, 2, 4\}$. Find the following sets:

$A \cup B =$ $\qquad\qquad$ $A \cap B =$

$A - B =$ $\qquad\qquad$ $B - A =$

$(A \cup B) - (A \cap B) =$ $\qquad\qquad$ $(A - B) \cup (B - A) =$

(b) Draw a Venn diagram to illustrate the statement P in general.

(c) Prove the statement P.

9. (a) [5] Let $B = \{2, 3, 5, 8\}$ and $C = \{3, 7\}$. Find $B \times C$ (that is, write out all the elements of this set).

(b) Prove that *for all sets* A*,* B*, and* C, if $A \subseteq B$, then $A \times C \subseteq B \times C$.

10. For each positive integer i, let $A_i = \left[-\frac{1}{i}, \frac{i}{i+1}\right]$. In the following, you need not prove that your answers are correct.

(a) Find $A_1 \cup A_2$ and $A_1 \cap A_2$.

(b) Find $\bigcup_{i \in \mathbb{Z}^+} A_i$ and $\bigcap_{i \in \mathbb{Z}^+} A_i$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 1 - x^3$ for all $x \in \mathbb{R}$.
(a) Prove that f is bijective.

(b) Find the inverse function f^{-1}.

12. Let X, Y, Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions.
(a) Prove that if $g \circ f$ is injective, then f is injective.

(b) Give an example of functions f and g for which $g \circ f$ is injective and g is not injective.

13. Let X, Y, Z be sets. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Prove that $g \circ f : X \to Z$ is invertible and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.