Math 365: Review Problems for Final Exam

The final exam will be cumulative. You will need to know everything from the lectures, homework, and old exams. The problems below are just a few representative problems, and are not meant to constitute a complete review.

1. Convert 2151_{nine} to base three without changing to base ten.

2. Calculate $1000_{\text{five}} - (122_{\text{five}} + 44_{\text{five}})$, without changing to base ten.

3. Find each the following, if it exists, and explain each answer.
   (a) $0 \div 1$ (b) $1 \div 0$ (c) $0 \div 0$

4. Find the following sums:
   (a) $4 + 5 + 6 + \cdots + 49$ (b) $4 + 7 + 10 + \cdots + 94$

5. Which of the following sets of ordered pairs are functions from the set of first components to the set of second components? (Circle those that are.)
   (a) $\{(a, 2),\ (b, 3),\ (b, 4),\ (c, 1)\}$
   (b) $\{(a, 1),\ (b, 1),\ (c, 2),\ (d, 3)\}$
   (c) $\{(a, 1),\ (b, 2),\ (c, 3),\ (d, 4)\}$

6. Construct a truth table for the proposition $p \longrightarrow q$.

7. Consider the following proposition about all whole numbers a, b, and d.

$$p: \text{ If } d|a \text{ and } d|b, \text{ then } d|(a+b).$$

   (a) Is p true? If not, give a counterexample.
   (b) State the *converse* of p. Is it true? If not, give a counterexample.
   (c) State the *contrapositive* of p. Is it true? If not, give a counterexample.

8. Consider the sets:

$$W = \{0, 1, 2, 3, \ldots\}, \quad I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \quad Q = \left\{\frac{a}{b} \mid a, b \in I \text{ and } b \neq 0\right\}.$$

State whether the following are true (T) or false (F).

(a) $W \subset I$ (b) $\frac{2}{5} \in Q$ (c) $I \cap W = I$ (d) W and Q are disjoint (e) $\sqrt{2} \in Q$

9. A jar contains pennies, nickels, and dimes. It contains three times as many pennies as nickels, and twice as many dimes as nickels.
(a) If the jar contains 4 nickels, what is the total value (in cents) of the coins in the jar?
(b) If the jar contains n nickels, what is the total value (in cents) of the coins in the jar (in terms of n)?

10. A survey of members of a health club found that:

24 members swim;

32 members use exercise bikes;

20 members use weight machines;

8 members swim and use weight machines;

13 members use exercise bikes and weight machines;

12 members use exercise bikes only;

5 members swim, use exercise bikes, and use weight machines;

6 members do not swim and do not use either exercise bikes or weight machines.

(a) How many members swim and use exercise bikes?
(b) How many members were surveyed?

11. Carla asks Phil to pick a number, add 17, and double the result. Then subtract 4, and double that result. Then add 20, then divide by 4, and finally subtract 20. When Phil told her the final result, she was able to tell him his original number. Explain how this trick works.

12. (a) In an arithmetic sequence, the seventh term minus the first term equals 9. The sum of the first and the seventh term is 5. Find the fourth term of the sequence.
(b) In a geometric sequence, the seventh term divided by the first term equals 64. The product of the first term and the seventh term is 16. Find the fourth term of the sequence.

13. Find the simplest form for each of the following:

(a) $\left(-\frac{1}{3}\right)^3 + 2^4 \div 5 \cdot \frac{1}{9} + 3^{-3}$ (b) $\frac{x^2 - xy}{y^2 - x^2}$

14. **(True/False/Counterexample)** For each statement, indicate whether it is true (T) or false (F). If it is false, give a counterexample.

(a) For all integers x and y, if $x^2 = xy$, then $x = y$.

(b) For all integers a, b, and d, if $d|a$ and $d|b$, then $\text{GCD}(a, b) = d$.

(c) For all natural numbers a, b, and c, $\frac{a}{ab + ac} = (b + c)^{-1}$.

(d) For all natural numbers a, b, and d, if $d|ab$, then $d|a$ or $d|b$.

15. Covert the following decimals to fractions:
(a) $0.\overline{31}$ (b) $42.1\overline{5}$

MATH 365 REVIEW PROBLEMS FOR FINAL EXAM : ANSWERS

1. $2151_{nine} = 2\cdot 9^3 + 1\cdot 9^2 + 5\cdot 9 + 1 = 2\cdot(3^2)^3 + 1\cdot(3^2)^2 + 5\cdot 3^2 + 1$
$= 2\cdot 3^6 + 1\cdot 3^4 + (3+2)\cdot 3^2 + 1$
$= 2\cdot 3^6 + 1\cdot 3^4 + 3\cdot 3^2 + 2\cdot 3^2 + 1 = 2\cdot 3^6 + 1\cdot 3^4 + 1\cdot 3^3 + 2\cdot 3^2 + 1$
$= 2011201_{three}$

2. $$\begin{array}{r} 122 \\ +\ 44 \\ \hline 221_{five} \end{array} \qquad \begin{array}{r} \overset{4\,4}{\overset{10\,10\,10}{\cancel{1}\cancel{0}\cancel{0}\cancel{0}}} \\ -\ 221 \\ \hline 224_{five} \end{array}$$

3. (a) $0 \div 1 = 0$ since 0 is the unique number c such that $1\cdot c = 0$
(b) $1 \div 0$ is undefined since there is no number c such that $0\cdot c = 1$
(c) $0 \div 0$ is undefined since there is not a unique number c such that $0\cdot c = 0$.

4. (a) $4+5+6+\cdots+49 = (1+2+3+4+5+6+\cdots+49) = \frac{49\cdot 50}{2} - (1+2+3) = 1225 - 6 = 1219$
(b) $\underbrace{4+\underbrace{7+10+\cdots+91}_{98}+94}_{98} = 98\cdot(\text{number of pairs})$
How many pairs are there? $94 = a_n = a_1 + (n-1)d = 4 + (n-1)\cdot 3$
so $90 = (n-1)3$, or $30 = n-1$, so $n = 31$
Answer: $98\cdot\frac{31}{2} = 1519$

5. (b), (c) are functions

6.

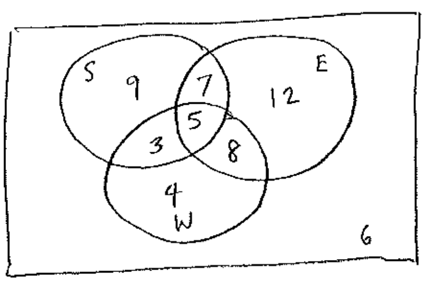
p	q	$p\to q$
T	T	T
T	F	F
F	T	T
F	F	T

7. (a) yes.
(b) If $d \mid (a+b)$, then $d \mid a$ and $d \mid b$. No.
Let $d=2$, $a=1$, $b=3$. Then
$2 \mid (1+3)$ but $2 \nmid 1$, $2 \nmid 3$.
(c) If $d \nmid (a+b)$, then $d \nmid a$ or $d \nmid b$. yes.

8. (a) T (b) T
(c) F (d) F (e) F

9. (a) 4 nickels, 12 pennies, 8 dimes: $4\cdot 5¢ + 12\cdot 1¢ + 8\cdot 10¢ = 20 + 12 + 80 = 112¢$
(b) n nickels, $3n$ pennies, $2n$ dimes: $n\cdot 5¢ + 3n\cdot 1¢ + 2n\cdot 10¢ = 5n + 3n + 20n = 28n$

10.



(a) $7+5 = 12$
(b) 54

11. Let x be Phil's number.
$$\frac{2(2(x+17)-4)+20}{4} - 20$$
$$= \frac{2(2x+34-4)+20}{4} - 20$$
$$= \frac{2(2x+30)+20}{4} - 20$$
$$= \frac{4x+80}{4} - 20 = \frac{4(x+20)}{4} - 20 = x$$
So Phil's final result is his original number.

12. (a) $a_7 - a_1 = 9$
$$\begin{array}{r} a_1 + a_7 = 5 \\ \hline 2a_7 = 14 \end{array}$$
$a_7 = 7$, $a_1 = -2$
$7 = a_7 = -2 + (7-1)d$
$9 = 6d$
$d = \frac{9}{6} = \frac{3}{2}$
$a_4 = -2 + (4-1)\cdot\frac{3}{2} = \frac{5}{2}$ or $2\frac{1}{2}$

(b) $\frac{a_7}{a_1} = 64$ $\quad a_1 a_7 = 16$
so $a_7 = 64a_1$
$a_1(64a_1) = 16$
$64a_1^2 = 16$
$a_1^2 = \frac{16}{64} = \frac{1}{4}$
so $a_1 = \frac{1}{2}$ or $-\frac{1}{2}$
$a_7 = 32$ or $-\frac{3}{2}$

If $a_1 = \frac{1}{2}$, $a_7 = 32$: $a_n = a_1 r^{n-1}$
$32 = \frac{1}{2}\cdot r^{7-1} = \frac{1}{2}r^6$ so
$64 = r^6$, or $r = 2$ or -2
If $r = 2$: $a_4 = \frac{1}{2}\cdot(2)^{4-1} = \frac{1}{2}\cdot 2^3 = \frac{1}{2}\cdot 8 = 4$
(Note: -4 is another possible answer.)

13. (a) $\left(-\frac{1}{3}\right)^3 = -\frac{1}{3^3} = -\frac{1}{27}$

$2^4 \div 5 \cdot \frac{1}{9} = \frac{2^4}{5} \cdot \frac{1}{9} = \frac{16}{5 \cdot 9} = \frac{16}{45}$

$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

So $\left(-\frac{1}{3}\right)^3 + 2^4 \div 5 \cdot \frac{1}{9} + 3^{-3} = -\frac{1}{27} + \frac{16}{45} + \frac{1}{27} = \frac{16}{45}$

(b) $\frac{x^2 - xy}{y^2 - x^2} = \frac{x(x-y)}{(y+x)(y-x)} = \frac{x(x-y)}{(y+x)(-1)(-y+x)} = \frac{x(x-y)}{(y+x)(-1)(x-y)} = \frac{x}{(y+x)(-1)} = -\frac{x}{x+y}$

14. (a) FALSE. Let $x=0$ and $y=1$. Then $0^2 = 0 \cdot 1$, but $0 \neq 1$.

(b) FALSE. Let $a=4$, $b=8$, and $d=2$. Then $2|4$ and $2|8$, but $GCD(4,8) = 4 \neq 2$.

(c) TRUE. $\left(\frac{a}{ab+ac} = \frac{a}{a(b+c)} = \frac{1}{b+c} = (b+c)^{-1} \right)$

(d) FALSE. Let $a=4$, $b=9$, and $d=6$. Then $6 | 4 \cdot 9$, but $6 \nmid 4$ and $6 \nmid 9$. (However it is true if d is a prime number.)

15. (a) Let $n = 0.\overline{31}$

$$\begin{aligned} 100n &= 31.\overline{31} \\ - \quad n &= 0.\overline{31} \\ \hline 99n &= 31 \\ n &= \frac{31}{99} \end{aligned}$$

(b) Let $n = 42.1\overline{5}$

$$\begin{aligned} 100n &= 4215.\overline{5} \\ - \quad 10n &= 421.\overline{5} \\ \hline 90n &= 3794 \\ n &= \frac{3794}{90} = \frac{1897}{45} \text{ or } 42\tfrac{7}{45} \end{aligned}$$

Alternatively:

$$\begin{aligned} 10n &= 421.5\overline{5} \\ - \quad n &= 42.1\overline{5} \\ \hline 9n &= 379.4 \\ n &= \frac{379.4}{9} = \frac{3794}{90} \end{aligned}$$

and continue as before.