Math 654 Homework Assignment 1

Due Friday, January 25

1. Let $V = \mathbb{R}^2$, and $T : V \to V$ be the linear transformation given by projection onto the y-axis. Then V is an $\mathbb{R}[x]$-module via $p(x) \cdot v = p(T)(v)$ $(v \in V)$. Find the $\mathbb{R}[x]$-submodules of V. Is V a cyclic $\mathbb{R}[x]$-module?

2. Let R be a ring, I an ideal, and M an R-module. Assume that $im = 0$ for all $i \in I$ and $m \in M$. Prove that M can be made into an R/I-module by defining $(r + I)m = rm$ for all $r \in R$, $m \in M$. (That is, show that this action of R/I on M is well-defined, and that M is indeed an R/I-module under this action.)

3. Let R be a ring and M an R-module. The *annihilator* of M in R is
$$\text{Ann}(M) = \{a \in R \mid am = 0 \text{ for all } m \in M\}.$$
   (a) Prove that $\text{Ann}(M)$ is a (two-sided) ideal of R.
   (b) Let V be a vector space over a field F, and $T : V \to V$ a linear transformation. Let V be an $F[x]$-module as in #1 above. Explain that $\text{Ann}(V)$ is the ideal generated by the minimal polynomial of T.
   (c) Let G be a finite abelian group (i.e. $\mathbb{Z}$-module). Explain that $\text{Ann}(G)$ is the ideal generated by $\exp(G)$, the exponent of G.

4. Let R be a ring, M an R-module, and N a submodule of M. If N and M/N are both finitely generated, prove that M is finitely generated.

5. Let $z \in R$ be an element in the center of R, that is, $zr = rz$ for all $r \in R$. Let $zM = \{zm \mid m \in M\}$.
   (a) Prove that zM is a submodule of M.
   (b) Let $e \in R$ be a *central idempotent*, that is, e is in the center of R and $e^2 = e$. Prove that $M = eM \oplus (1 - e)M$.