Math 654 Homework Assignment 2

Due Friday, February 1

Let R be a ring (with 1), and M an R-module (understood to be unitary). Define

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\},$$

the set of *torsion* elements of M. We say M is a *torsion R-module* if $\mathrm{Tor}(M) = M$.

1. (a) Let G be a finitely generated abelian group (i.e. $\mathbb{Z}$-module). Describe $\mathrm{Tor}(G)$.
   (b) If R is an integral domain, prove that $\mathrm{Tor}(M)$ is a submodule of M.

2. A nonzero R-module M is *simple* (or *irreducible*) if $\{0\}$ and M are its only submodules.
   (a) Find all simple $\mathbb{Z}$-modules.
   (b) Prove that M is simple if, and only if, M is a cyclic module generated by any of its nonzero elements.
   (c) Assume R is commutative. Prove that M is irreducible if, and only if, M is isomorphic (as an R-module) to R/I for some maximal ideal I of R.

3. Prove *Schur's Lemma*: Let R be a ring and M and N be two simple R-modules. Then every nonzero element of $\mathrm{Hom}_R(M, N)$ is an isomorphism. In particular, $\mathrm{End}_R(M)$ is a division ring and $\mathrm{Hom}_R(M, N) = 0$ if $M \not\cong N$. (Hint: Consider the kernel and image of each homomorphism.)

4. Show by verifying the following example that for some rings, rank of a free module is not uniquely defined: Let V be a vector space over a field F with countably infinite basis $\{v_1, v_2, v_3, \ldots\}$. Let $R = \mathrm{End}_F(V)$, the ring of all F-linear transformations from V to V. Show that there is an R-module isomorphism $\phi : R \to R \oplus R$. (Hint: Set $\phi(T) = (T \circ s_1, T \circ s_2)$ and $\psi((T, U)) = T \circ s_1' + U \circ s_2'$ for all $T \in R$, where s_1, s_2, s_1', s_2' are linear transformations defined by $s_1(v_n) = v_{2n}$, $s_2(v_n) = v_{2n-1}$, $s_1'(v_{2n}) = v_n$, $s_1'(v_{2n-1}) = 0$, $s_2'(v_{2n}) = 0$, $s_2'(v_{2n-1}) = v_n$ for all $n \geq 1$. Check that ϕ, ψ are inverse maps.)

5. Assume R is a *commutative* ring. Let M, M', N, N' be R-modules, and let $f : M \to M'$, $g : N \to N'$ be R-module homomorphisms.
   (a) Show that (left) composition with g induces an R-module homomorphism from $\mathrm{Hom}_R(M, N)$ to $\mathrm{Hom}_R(M, N')$.
   (b) Show that (right) composition with f induces an R-module homomorphism from $\mathrm{Hom}_R(M', N)$ to $\mathrm{Hom}_R(M, N)$.