Math 654 Homework Assignment 3

Assume R is a *commutative* ring (with 1). If M is an R-module, recall

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

We say M is a *torsion R-module* if $\mathrm{Tor}(M) = M$.

1. Let A be any abelian group and let m, n be positive integers greater than 1.
   (a) Show that
   $$\mathbb{Z}_m \otimes_{\mathbb{Z}} A \cong A/mA \quad \text{and} \quad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, A) \cong A[m]$$
   where $A[m]$ is the subgroup $\{a \in A \mid ma = 0\}$ of A.
   (b) Use part (a) to show that
   $$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)} \quad \text{and} \quad \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)},$$
   where (m, n) is the greatest common divisor of m and n.

2. Let $\mathbb{Q}$ denote the (additive) group of rational numbers.
   (a) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.
   (b) Let A be a torsion abelian group (i.e. torsion $\mathbb{Z}$-module). Show that $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

3. Let M and N be R-modules. Prove that $M \otimes_R N \cong N \otimes_R M$.

4. Let M, M', N, N' be R-modules, and let $f : M \to M'$, $g : N \to N'$ be R-module homomorphisms. Prove that there is a unique R-module homomorphism from $M \otimes_R N$ to $M' \otimes_R N'$ (denoted $f \otimes g$) such that $m \otimes n$ is mapped to $f(m) \otimes g(n)$ for all $m \in M, n \in N$.

5. Let G be a finitely generated abelian group. Prove that $G \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r$ for some nonnegative integer r.