Math 654 Homework Assignment 4

Due Friday, February 22

Let R be a ring with 1.

1. Suppose R is commutative. Let M a free R-module of finite rank. Let $f, g : M \to M$ be R-module homomorphisms.
   (a) Prove that $\det(f \circ g) = \det(f)\det(g)$.
   (b) Prove that if f is bijective, then $\det(f)$ is an invertible element of R. (In fact, the converse is also true; see Hungerford Proposition VII.3.7.)

2. Prove that a direct sum $\oplus_{i \in I} P_i$ of R-modules is projective if, and only if, P_i is projective for all $i \in I$.

3. Suppose R is a principal ideal domain. Prove that every finitely generated projective R-module is free. (Hint: Use the classification of finitely generated modules over a PID.)

4. Let $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ be a short exact sequence of R-modules. Prove that it is split if, and only if, there is an R-module homomorphism $\delta : V \to U$ such that $\delta \circ \alpha = \mathrm{id}_U$.