Math 654 Homework Assignment 5

Due Friday, March 1

Let R be a ring with 1. All modules are assumed to be unitary.

1. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Prove that under the induced maps, the sequence
$$0 \to \mathrm{Hom}_R(C, D) \xrightarrow{\beta^*} \mathrm{Hom}_R(B, D) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A, D)$$
is exact for any R-module D. (Here, for example, $\beta^*(f) = f \circ \beta$.)

2. Suppose R is commutative. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Prove that under the induced maps, the sequence
$$A \otimes_R D \xrightarrow{\alpha_*} B \otimes_R D \xrightarrow{\beta_*} C \otimes_R D \to 0$$
is exact for any R-module D. (Here, for example, $\alpha_* = \alpha \otimes_R \mathrm{id}_D$.) *(Hint: Look in Hungerford.)*

3. Prove that any direct summand of an injective R-module I is injective.

4. Suppose that R is an integral domain. Let S be a multiplicative subset of R such that $0 \notin S$. Prove that $S^{-1}R$ is isomorphic to a subring of the field of fractions of R.

5. Suppose that R is commutative. Let P be a prime ideal of R, and let $S = R - P$.
   (a) Prove that S is a multiplicative set.
   (b) Prove that $S^{-1}R$ has a unique maximal ideal. (Terminology and notation: $S^{-1}R$ is called the *localization* of R at P, and is denoted R_P. In general, a commutative ring having a unique maximal ideal is called a *local* ring.)