Math 654 Homework Assignment 6

Due Friday, March 29

1. Let F be a field. Prove that F contains a unique smallest subfield F_0 that is isomorphic either to $\mathbb{Q}$ or to $\mathbb{F}_p$ for some prime p. (Hint: Recall the definition of the characteristic of a ring, and note that the characteristic of a field must either be 0 or prime. Terminology: The field F_0 is called the *prime subfield* of F.)

2. Let $D \in \mathbb{Z} - \{0\}$, D not a square in $\mathbb{Z}$, and let $\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}$, a subring of $\mathbb{R}$ or $\mathbb{C}$. Define $\phi : \mathbb{Z}[\sqrt{D}] - \{0\} \to \mathbb{Z}$ by
$$\phi\left(a + b\sqrt{D}\right) = a^2 - b^2 D.$$
   (a) Let $\alpha, \beta \in \mathbb{Z}[\sqrt{D}]$. Prove that $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$.
   (b) Prove that $\alpha \in \mathbb{Z}[\sqrt{D}]$ is a unit in $\mathbb{Z}[\sqrt{D}]$ if, and only if, $\phi(\alpha) = \pm 1$.
   (c) Determine the units in $\mathbb{Z}[i]$. (Here $i = \sqrt{-1}$.)
   (d) Let $p \in \mathbb{Z}$ be prime. If $\alpha \in \mathbb{Z}[\sqrt{D}]$ and $\phi(\alpha) = \pm p$, prove that α is irreducible in $\mathbb{Z}[\sqrt{D}]$.
   (e) Show that $1 + i$, $1 - i$, and 3 are irreducible in $\mathbb{Z}[i]$. Conclude that $1 + i$, $1 - i$, and 3 are prime in $\mathbb{Z}[i]$ but that 2 is not a prime in $\mathbb{Z}[i]$. (You may use the result of #3 below, that $\mathbb{Z}[i]$ is a Euclidean domain.)
   (f) Show that $2 + \sqrt{5}i$ is irreducible in $\mathbb{Z}[\sqrt{5}i]$ but is not prime. Conclude that $\mathbb{Z}[\sqrt{5}i]$ is an integral domain that is not a unique factorization domain.

3. Prove that $\mathbb{Z}[i]$ is a Euclidean domain, with norm ϕ given in #2 above, as follows.
   (a) Let $\alpha, \beta \in \mathbb{Z}[i]$, $\beta \neq 0$. Show that, as an element of $\mathbb{C}$, $\alpha\beta^{-1} = r + si$ for some $r, s \in \mathbb{Q}$.
   (b) In part (a), if $r, s \in \mathbb{Z}$, then $\alpha = (r + si)\beta$ with $r + si \in \mathbb{Z}[i]$. If not, let p be an integer closest to r and q an integer closest to s. Let $\theta = (r-p)+(s-q)i$ and $\gamma = \theta\beta$. (Note that $\gamma \in \mathbb{Z}[i]$ since $\gamma = \alpha - (p + qi)\beta$.) Show that
$$\phi(\theta) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad \text{and} \quad \phi(\gamma) \leq \frac{1}{2}\phi(\beta).$$
   It follows that $\alpha = (p + qi)\beta + \gamma$ with $\gamma = 0$ or $\phi(\gamma) < \phi(\beta)$.

4. Let R be an integral domain, $r \in R$, $r \neq 0$, and let S be the set of all proper principal ideals of R. Prove that r is irreducible if and only if the ideal (r) is maximal in the set S.

5. Let $f(x) = x^5 + 6x^4 + 9x^2 - 12 \in \mathbb{Z}[x]$. Use Eisenstein's Criterion to prove that f is irreducible in both $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$.