Math 654 Homework Assignment 7

Due Friday, April 5

1. Let $p(x) = x^3 - 6x^2 + 9x + 3 \in \mathbb{Q}[x]$, and let α be any root of $p(x)$ in $\mathbb{C}$. Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$, and a basis of $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$. Express α^4 as a linear combination of your basis elements.

2. Find $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ and find a basis of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ over $\mathbb{Q}$. Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

3. Let K/F be a field extension and $\alpha \in K$ such that $[F(\alpha) : F]$ is odd. Prove that $F(\alpha) = F(\alpha^2)$.

4. Let $f(x)$ be an irreducible polynomial of degree n over a field F. Let $g(x) \in F[x]$. Prove that every irreducible factor of $f(g(x))$ has degree divisible by n.

5. Let K_1 and K_2 be finite extensions of a field F, both contained in a field K.
   (a) Prove that $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$.
   (Hint: Let $\alpha_1, \ldots, \alpha_m$ be a basis for K_1 over F, and $\beta_1, \ldots, \beta_n$ be a basis for K_2 over F. Write $K_1 = F(\alpha_1, \ldots, \alpha_m)$ and $K_2 = F(\beta_1, \ldots, \beta_n)$.)
   (b) If $[K_1 : F] = m$ and $[K_2 : F] = n$ with $(n, m) = 1$, prove that $[K_1K_2 : F] = mn$.