

Math 654 Homework Assignment 9

Due Friday, April 26

1. Let K be the splitting field of $x^4 - 4$ over \mathbb{Q} . Let $G = \text{Gal}(K/\mathbb{Q})$.
 - (a) Describe K as an extension of \mathbb{Q} by one or two elements.
 - (b) Determine G and find all subgroups of G and their fixed fields.
2. Show that $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$ is a Galois extension of \mathbb{Q} , and determine its Galois group.
3. Let $a, b, c \in \mathbb{Z}$, $a \neq 0$, and let K be the splitting field of $ax^2 + bx + c$ over \mathbb{Q} . Determine all possible Galois groups $G = \text{Gal}(K/\mathbb{Q})$, and give conditions on a, b, c under which each group occurs.
4. Let K and L be two finite Galois extensions of a field F .
 - (a) Prove that $K \cap L$ is Galois over F .
 - (b) Prove that KL is Galois over F .
5. Let p be a prime. Let K/F be a Galois extension of degree p^n for some positive integer n . Prove that there are Galois extensions of F contained in K of all possible degrees, $1, p, p^2, \dots, p^{n-1}, p^n$.
6. Prove that every finite group occurs as the Galois group of *some* field extension. (Compare with the *inverse Galois problem* (unsolved), that of determining which finite groups arise as Galois groups of extensions of \mathbb{Q} .) (*Hint: Let F be any field and $K = F(x_1, \dots, x_n)$, the field of rational functions in indeterminates x_1, \dots, x_n . Let s_1, \dots, s_n denote the elementary symmetric polynomials, and $E = F(s_1, \dots, s_n)$. Then K/E is Galois with Galois group S_n , the symmetric group of degree n , by a result of Emil Artin. Now use Cayley's Theorem and the Fundamental Theorem of Galois Theory.*)