Math 300 Exam 2
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Name Solutions

There are 6 questions, for a total of 100 points. Point values are written beside each question.

1. [15 points] Prove by induction that for all positive integers n,

$$P(n): \quad \sum_{i=0}^{n-1} 4^i = \frac{1}{3}(4^n - 1).$$

1. Check that $P(1)$ is true:

$$\sum_{i=0}^{1-1} 4^i = \sum_{i=0}^{0} 4^i = 4^0 = 1$$

$$\frac{1}{3}(4^1 - 1) = \frac{1}{3} \cdot 3 = 1$$

2. Assume that $P(k)$ is true for some positive integer k, i.e. (induction hypothesis)

$$\sum_{i=0}^{k-1} 4^i = \frac{1}{3}(4^k - 1).$$

3. Prove that $P(k+1)$ is true:

$$\sum_{i=0}^{(k+1)-1} 4^i = \sum_{i=0}^{k} 4^i = 4^0 + 4^1 + \cdots + 4^{k-1} + 4^k$$

$$= \frac{1}{3}(4^k - 1) + 4^k \quad \text{(by the induction hypothesis)}$$

$$= \frac{1}{3} \cdot 4^k - \frac{1}{3} + 4^k$$

$$= 4^k\left(\frac{1}{3} + 1\right) - \frac{1}{3}$$

$$= 4^k\left(\frac{1}{3} + \frac{3}{3}\right) - \frac{1}{3}$$

$$= 4^k \cdot \frac{4}{3} - \frac{1}{3}$$

$$= \frac{4^{k+1}}{3} - \frac{1}{3}$$

$$= \frac{1}{3}(4^{k+1} - 1).$$

Therefore $P(k+1)$ is true. 1 By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

2. [15] Let $a_1 = 1$, $a_2 = 7$, and $a_n = 7a_{n-1} - 12a_{n-2}$ for all positive integers $n \geq 3$. Prove that for all positive integers n, $a_n = 4^n - 3^n$.

$P(n)$: $a_n = 4^n - 3^n$

1. Check $P(1)$ is true: $a_1 = 1$

$4^1 - 3^1 = 4 - 3 = 1$

2. Assume that for some integer k, $a_i = 4^i - 3^i$ for all integers i for which $1 \leq i \leq k$.

3. Prove that $a_{k+1} = 4^{k+1} - 3^{k+1}$:

By definition,

$$
\begin{aligned}
a_{k+1} &= 7a_k - 12a_{k-1} \\
&= 7(4^k - 3^k) - 12(4^{k-1} - 3^{k-1}) \\
&= 7\cdot 4^k - 7\cdot 3^k - 12\cdot 4^{k-1} + 12\cdot 3^{k-1} \\
&= 7\cdot 4^k - 7\cdot 3^k - 3\cdot 4^k + 4\cdot 3^k \\
&= 7\cdot 4^k - 3\cdot 4^k - 7\cdot 3^k + 4\cdot 3^k \\
&= 4\cdot 4^k - 3\cdot 3^k \\
&= 4^{k+1} - 3^{k+1}.
\end{aligned}
$$

Therefore $P(k+1)$ is true. By the Second Principle of Mathematical Induction, $P(n)$ is true for all positive integers n.

3. Consider the following two sets:

$$A = \{n \in \mathbb{Z} \mid n = 3r - 1 \text{ for some } r \in \mathbb{Z}\}$$
$$B = \{n \in \mathbb{Z} \mid n = 3s + 5 \text{ for some } s \in \mathbb{Z}\}$$

(a) [5] List at least 5 elements of A and at least 5 elements of B.

A: $-1, 2, 5, 8, 11$

B: $5, 8, 11, 14, 17$

(b) [5] Is $A \subseteq B$? Prove or disprove.

Yes. Let $n \in A$, so that $n = 3r - 1$ for some $r \in \mathbb{Z}$. Then

$$\begin{aligned} n &= 3r - 1 \\ &= 3r + 5 - 5 - 1 \\ &= (3r - 6) + 5 \\ &= 3(r-2) + 5. \end{aligned}$$

Let $s = r - 2$. Then $n = 3s + 5$ so $n \in B$ since $s \in \mathbb{Z}$.
Therefore $A \subseteq B$.

(c) [5] Is $B \subseteq A$? Prove or disprove.

Yes. Let $n \in B$, so that $n = 3s + 5$ for some $s \in \mathbb{Z}$.
Then

$$\begin{aligned} n &= 3s + 5 \\ &= 3s - 1 + 1 + 5 \\ &= (3s + 6) - 1 \\ &= 3(s+2) - 1 \end{aligned}$$

Let $r = s + 2$. Then $n = 3r - 1$, which is an element of A since $r \in \mathbb{Z}$.
Therefore $B \subseteq A$.

4. Consider the following statement.

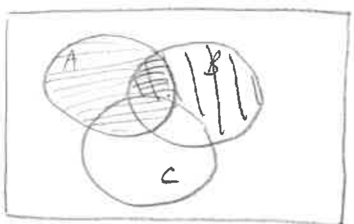
P: For all sets A, B, and C, $A \cap (B - C) = (A - C) \cap B$.

(a) [4] *Just for this part,* let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, and $C = \{3, 4, 5, 6\}$. Find the following sets:

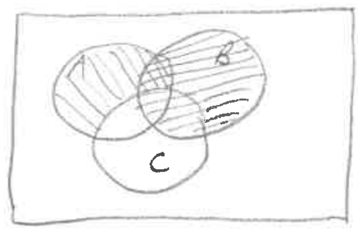
$B - C = \{2\}$ $\quad\quad$ $A - C = \{1, 2\}$

$A \cap (B - C) = \{2\}$ $\quad\quad$ $(A - C) \cap B = \{2\}$

(b) [4] Draw Venn diagram(s) to illustrate the statement P in general.



$A \cap (B - C)$



$(A - C) \cap B$

(c) [7] Prove the statement P.

$$
\begin{aligned}
A \cap (B - C) &= A \cap (B \cap C^c) \\
&= A \cap B \cap C^c \\
&= A \cap C^c \cap B \\
&= (A \cap C^c) \cap B \\
&= (A - C) \cap B
\end{aligned}
$$

(This can also be proven element wise.)

5. Let $f : \mathbb{R} \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be functions given by

$$f(x) = (x, -x) \quad \text{and} \quad g(x, y) = x - y$$

for all $x, y \in \mathbb{R}$.

(a) [6] What are all preimages of $(0, 0)$ for f? What are all preimages of 0 for g?

$(x, -x) = (0, 0)$

$x = 0$

preimages of $(0,0)$: 0

or in set notation, $\{0\}$

$x - y = 0$

$x = y$

preimages of 0: all (x, y) with $x = y$,

i.e.

$\{(x, x) \mid x \in \mathbb{R}\}$

(b) [6] Find $f \circ g$ and $g \circ f$.

$(f \circ g)(x, y) = f(g(x,y)) = f(x-y) = (x-y, -(x-y))$

or $(x-y, y-x)$

$(g \circ f)(x) = g(f(x)) = g(x, -x) = x - (-x) = 2x$

(c) [8] Answer the following questions (*yes* or *no*). You need not justify your answers.

Is f injective? Yes

Is f surjective? No

Is g injective? No

Is g surjective? Yes

6. [20] *(True/False/Counterexample.)* For each statement, determine whether it is true or false, and accordingly write "T" or "F" in the blank. If the statement is false, provide a counterexample. (No need to prove true statements.)

__T__ For all sets A and B, $A \subseteq B$ if and only if $B \cap A = A$.

__F__ For all subsets A and B of a universal set U, $A^c \subseteq B^c$ if and only if $A \subseteq B$.

Counterexample: $A = \{1,2\}$, $B = \{1,2,3\}$, $U = \{1,2,3\}$. Then $A \subseteq B$.

$A^c = \{3\}$ $\quad B^c = \phi$ $\quad$ So $A^c \nsubseteq B^c$.

__T__ Each set consisting of three elements has exactly eight subsets.

__F__ For all sets A and invertible functions $f : A \to A$ and $g : A \to A$,

$$(g \circ f)^{-1} = g^{-1} \circ f^{-1}.$$

Counterexample: $f : \mathbb{R} \to \mathbb{R}$, $\quad g : \mathbb{R} \to \mathbb{R}$

$f(x) = 2x$ $\quad g(x) = x+1$

$f^{-1}(x) = \frac{1}{2}x$ $\quad g^{-1}(x) = x-1$

$(g \circ f)(x) = g(2x) = 2x+1$ $\quad (g \circ f)^{-1}(x) = \frac{1}{2}(x-1)$

$(g^{-1} \circ f^{-1})(x) = g^{-1}(\frac{1}{2}x) = \frac{1}{2}x - 1$

__F__ For all sets A and functions $f : A \to A$, if f is surjective, then f is injective.

Counterexample: $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ $\quad$ (where $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$)

$f(x) = x^2$

Then f is surjective.

However, $f(1) = f(-1)$ so f is not injective.