MATH 662 SPRING 2025
BRIEF COURSE NOTES

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These notes will include some, but not all, of the material from class. In particular, proofs of theorems and examples are more likely to appear in class than in these notes. Some of these notes will be adapted and expanded from the appendix of [2].

Throughout, R will be a ring with $1 \neq 0$. Each R-module M is assumed to be *unital*, i.e. the multiplicative identity 1 of R acts as the identity map on M. We will work with both left and right modules, and where this distinction is essential, it will be specified which one.

1. Complexes

A *complex* $C_\bullet$ (or $(C_\bullet, d_\bullet)$ or (C, d)) of R-modules is a sequence of R-modules and R-module homomorphisms (called *differentials*),

$$C_\bullet : \qquad \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

for which $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The *degree* $|x|$ of an element x of C_n is n. Under this terminology, each of the differentials d_n has degree -1 as a map.

For each degree n, we define R-submodules of C_n and a subquotient as follows.

$$\begin{aligned} Z_n(C_\bullet) &= \operatorname{Ker}(d_n) \quad \text{(the } n\text{-cycles)} \\ B_n(C_\bullet) &= \operatorname{Im}(d_{n+1}) \quad \text{(the } n\text{-boundaries)} \\ H_n(C_\bullet) &= Z_n(C_\bullet)/B_n(C_\bullet) \quad \text{(the } n\text{th homology)} \end{aligned}$$

We say that two n-cycles x and y are *homologous* if $x - y$ is an n-boundary, that is, $x - y \in B_n$. We collect all the homology modules together and write

$$\mathrm{H}_*(C_\bullet) = \oplus_{n\in\mathbb{Z}} \mathrm{H}_n(C_\bullet),$$

the *homology* of $C_\bullet$ (or of C, omitting the subscript for simplicity of notation). It is common to identify C with the R-module $\oplus_{n\in\mathbb{Z}} C_n$, and d with the endomorphism of this direct sum that is just d_n on each C_n as identified canonically with an R-submodule of this direct sum.

Some further terminology (that is not universally agreed upon): A *chain complex* is a complex for which $C_n = 0$ for $n < 0$. A *cochain complex* is a complex for which $C_n = 0$ for $n > 0$. These terms are also used more generally in the literature to refer to complexes as we have defined them here.

We may wish to index complexes differently, replacing n by $-n$ in $C_\bullet$ above, with the maps still oriented as shown. Then the indexing in the above diagram is visually the same as the ordering of integers on a number line. A cochain complex then has differential of degree +1; the index is often then written as a superscript:

$$C^{\bullet}: \qquad 0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

With this indexing, elements in the kernel of d_n are called the *n-cocycles*, and elements in the image of d_{n-1} are called the *n-coboundaries*. Two n-cocycles are called *cohomologous* if their difference is an n-coboundary. Similar to the above, we set

$$\mathrm{H}^n(C^{\bullet}) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n-1})$$

and $\mathrm{H}^*(C^{\bullet}) = \oplus_{n\geq 0}\,\mathrm{H}^n(C^{\bullet})$, the *cohomology* of the cochain complex $C^{\bullet}$.

A complex $C_{\bullet}$ is called *acyclic*, or *exact*, if $\mathrm{H}_n(C_{\bullet}) = 0$ for all n. A *short exact sequence* is an exact complex of the form $0 \to U \to V \to W \to 0$.

Let (C, d) and (C', d') be complexes. A *chain map* $f_{\bullet} : C_{\bullet} \to C'_{\bullet}$ is a collection of R-module homomorphisms $f_n : C_n \to C'_n$ for which $f_{n-1}d_n = d'_n f_n$ for each $n \in \mathbb{Z}$. That is, the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \\
\cdots & \longrightarrow & C'_1 & \xrightarrow{d'_1} & C'_0 & \xrightarrow{d'_0} & C'_{-1} & \longrightarrow & \cdots
\end{array}$$

It can be checked that a chain map induces a map on homology. A chain map is called a *quasi-isomorphism* if this induced map is an isomorphism on homology.

We say that two chain maps $f_{\bullet}, g_{\bullet} : C_{\bullet} \to C'_{\bullet}$ are *chain homotopic* if there exist R-module homomorphisms $s_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = s_{n-1}d_n + d'_{n+1}s_n \tag{1.1}$$

for all n. The collection $s_{\bullet}$ of homomorphisms is called a *homotopy* for $f_{\bullet} - g_{\bullet}$. It can be checked that chain homotopy is an equivalence relation, and that two chain homotopic maps induce the same maps on homology. As a special case, when $g_{\bullet}$ is the zero map, we call $s_{\bullet}$ a *chain contraction* of $f_{\bullet}$. A chain contraction of the identity map on $C_{\bullet}$, if it exists, is sometimes called a *contracting homotopy*, and in this case, it can be checked that $C_{\bullet}$ is acyclic. (In fact, for this last consequence, it is not needed that the functions s_n are R-module homomorphisms, only that there are such functions (of sets, or of abelian groups, for example) satisfying equation (1.1).)

2. Projective resolutions

We call an R-module P *projective* if for every surjective R-module homomorphism $f : U \to V$ and R-module homomorphism $g : P \to V$, there exists an R-module homomorphism $h : P \to U$ such that $fh = g$:

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle g} & & \\
U & \xrightarrow{f} & V & \longrightarrow & 0
\end{array} \tag{2.1}$$

There are other equivalent definitions of projective module. For example, an R-module is projective if, and only if, it is a direct summand of a free module (i.e. $R^{\oplus I}$ for some indexing set I).

Let M be an R-module. A *projective resolution* of M is a chain complex $P_\bullet$ consisting of projective R-modules P_n $(n \geq 0)$ for which $\mathrm{H}_0(P_\bullet) \cong M$ and $\mathrm{H}_n(P_\bullet) = 0$ for all $n \neq 0$. As a consequence, $P_\bullet$ is quasi-isomorphic to the complex that is M concentrated in degree 0 and 0 elsewhere:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Another consequence of the definition is that the following sequence is exact:

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0. \tag{2.2}$$

(In some texts, (2.2) is called the projective resolution of M.) The complex (2.2) is sometimes called the *augmented complex* of $P_\bullet$. This augmented complex may be abbreviated $P_\bullet \xrightarrow{\varepsilon} M$. The complex $P_\bullet$ (without the M) is sometimes called the *truncated complex* of (2.2).

Projective resolutions of R-modules always exist: Every R-module M is a homomorphic image of a projective R-module, for example, the free module on a set of generators of M. One may use this fact to build a projective resolution as follows. Let P_0 be a projective R-module mapping surjectively to M via an R-module homomorphism ε. Let $K_1 = \mathrm{Ker}(\varepsilon)$. In turn, K_1 is a homomorphic image of some projective R-module P_1 via some R-module homomorphism $\varepsilon_1 : P_1 \to K_1$. Denote by i_1 the inclusion map $i_1 : K_1 \to P_0$ and set $d_1 = i_1\varepsilon_1$. Let $K_2 = \mathrm{Ker}(d_1)$ and continue. Visually, we have:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_2 & & \xrightarrow{\quad d_2 \quad} & & P_1 & & \xrightarrow{\quad d_1 \quad} & & P_0 & \xrightarrow{\varepsilon} M \longrightarrow 0 \\ & & & {\scriptstyle \varepsilon_2}\searrow & & \nearrow{\scriptstyle i_2} & & {\scriptstyle \varepsilon_1}\searrow & & \nearrow{\scriptstyle i_1} & & \\ & & & & K_2 & & & & K_1 & & & \end{array} \tag{2.3}$$

The R-module K_i is called an *ith syzygy module* of M. This module depends on some choices. However, it is unique up to an equivalence relation, as stated in Lemma 2.5 below. We will first need Schanuel's Lemma:

Lemma 2.4 (Schanuel's Lemma)**.** *Let* $0 \to K \to P \xrightarrow{\varepsilon} M \to 0$ *and* $0 \to K' \to P' \xrightarrow{\varepsilon'} M \to 0$ *be two short exact sequences of R-modules with P, P' projective. Then* $K \oplus P' \cong K' \oplus P$.

A proof is given in class.

The next lemma is a consequence of Schanuel's Lemma via a mathematical induction argument.

Lemma 2.5. *Let K_i and K'_i be two ith syzygy modules of the R-module M. There are projective R-modules P, P' such that* $K_i \oplus P \cong K'_i \oplus P'$.

A proof is given in class.

Remark 2.6. There is another way to state Lemma 2.5: Call two R-modules U and V *equivalent* if there exist projective R-modules P, P' for which $U \oplus P \cong V \oplus P'$ as R-modules. This can be shown to be an equivalence relation. The conclusion of Lemma 2.5 is that K_i and K'_i are equivalent under this equivalence relation.

The next theorem implies a relation among projective resolutions themselves.

Theorem 2.7 (Comparison Theorem). *Let $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ be chain complexes of R-modules with $M = \mathrm{H}_0(P_\bullet)$, $N = \mathrm{H}_0(Q_\bullet)$, and let $\varepsilon : P_0 \to M$ and $\varepsilon' : Q_0 \to N$ be corresponding augmentation maps. Assume that P_i is projective for each i and that the augmented complex $\cdots \to Q_1 \to Q_0 \to N \to 0$ is exact. If $f : M \to N$ is an R-module homomorphism, then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ for which $f\varepsilon = \varepsilon' f_0$, that is, the following diagram commutes:*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\varepsilon'} & N & \longrightarrow & 0 \end{array}$$

The chain map $f_\bullet$ is unique up to chain homotopy.

As a consequence of the theorem, if $P_\bullet, Q_\bullet$ are projective resolutions of M, N, respectively, the Comparison Theorem states that there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ *lifting* the R-module homomorphism $f : M \to N$.

References

[1] C. A. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.

[2] S. Witherspoon, *Hochschild cohomology for algebras*, Graduate Studies in Mathematics, vol. 204, American Mathematical Society, 2019.

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