MATH 662 SPRING 2025
BRIEF COURSE NOTES

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These notes include some, but not all, of the material from class. Proofs of theorems and examples are more likely to appear in class than in these notes. The appendix includes some supplementary topics that may not be covered in class.

Throughout, R will be a ring with $1 \neq 0$. Each R-module M is assumed to be *unital*, i.e. the multiplicative identity 1 of R acts as the identity map on M. We will work with both left and right modules, and where this distinction is essential, it will be specified which one.

1. Complexes

A *complex* $C_\bullet$ (or $(C_\bullet, d_\bullet)$ or (C, d)) of R-modules is a sequence of R-modules and R-module homomorphisms (called *differentials*),

$$C_\bullet : \qquad \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

for which $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The *degree* $|x|$ of an element x of C_n is n. Under this terminology, each of the differentials d_n has degree -1 as a map.

For each degree n, we define R-submodules and a subquotient of C_n as follows.

$$\begin{aligned} Z_n(C_\bullet) &= \mathrm{Ker}(d_n) && \text{(the } n\text{-cycles)} \\ B_n(C_\bullet) &= \mathrm{Im}(d_{n+1}) && \text{(the } n\text{-boundaries)} \\ H_n(C_\bullet) &= Z_n(C_\bullet)/B_n(C_\bullet) && \text{(the } n\text{th homology)} \end{aligned}$$

We say that two n-cycles x and y are *homologous* if $x - y$ is an n-boundary, that is, $x - y \in B_n$. We collect all the homology modules together and write

$$\mathrm{H}_*(C_\bullet) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_n(C_\bullet),$$

the *homology* of $C_\bullet$ (or of C, omitting the subscript for simplicity of notation). It is common to identify C with the R-module $\oplus_{n\in\mathbb{Z}} C_n$, and d with the endomorphism of this direct sum that is just d_n on each C_n as identified canonically with an R-submodule of this direct sum.

Some further terminology (that is not used universally or consistently in the literature): A *chain complex* is a complex for which $C_n = 0$ for $n < 0$. A *cochain complex* is a complex for which $C_n = 0$ for $n > 0$. These two terms are also used more generally in the literature to refer to complexes as we have defined them here.

We may wish to index complexes differently, replacing n by $-n$ in $C_\bullet$ above, with the maps still oriented as shown. Then the indexing in the above diagram is visually the same as the ordering of integers on a number line. A cochain complex then has differential of degree +1; the index is often then written as a superscript:

$$C^\bullet : \qquad 0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

With this indexing, elements in the kernel of d_n are called the *n-cocycles*, and elements in the image of d_{n-1} are called the *n-coboundaries*. Two n-cocycles are called *cohomologous* if their difference is an n-coboundary. Similar to the above, we set

$$\mathrm{H}^n(C^\bullet) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n-1})$$

and $\mathrm{H}^*(C^\bullet) = \oplus_{n\geq 0} \mathrm{H}^n(C^\bullet)$, the *cohomology* of the cochain complex $C^\bullet$.

A complex $C_\bullet$ is called *acyclic*, or *exact*, if $\mathrm{H}_n(C_\bullet) = 0$ for all n. A *short exact sequence* is an exact complex of the form $0 \to U \to V \to W \to 0$.

Let (C, d) and (C', d') be complexes. A *chain map* $f_\bullet : C_\bullet \to C'_\bullet$ is a collection of R-module homomorphisms $f_n : C_n \to C'_n$ for which $f_{n-1}d_n = d'_n f_n$ for each $n \in \mathbb{Z}$. That is, the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \\
\cdots & \longrightarrow & C'_1 & \xrightarrow{d'_1} & C'_0 & \xrightarrow{d'_0} & C'_{-1} & \longrightarrow & \cdots
\end{array}$$

It can be checked that a chain map induces a map on homology. A chain map is called a *quasi-isomorphism* if this induced map is an isomorphism on homology.

We say that two chain maps $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ are *chain homotopic* if there exist R-module homomorphisms $s_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = s_{n-1}d_n + d'_{n+1}s_n \tag{1.1}$$

for all n. The collection $s_\bullet$ of homomorphisms is called a *homotopy* for $f_\bullet - g_\bullet$. It can be checked that chain homotopy is an equivalence relation, and that two chain homotopic maps induce the same maps on homology. As a special case, when $g_\bullet$ is the zero map, we call $s_\bullet$ a *chain contraction* of $f_\bullet$. A chain contraction of the identity map on $C_\bullet$, if it exists, is sometimes called a *contracting homotopy*, and in this case, it can be checked that $C_\bullet$ is acyclic. (In fact, for this last consequence, it is not needed that the functions s_n are R-module homomorphisms, only that there are such functions (of sets, or of abelian groups, for example) satisfying equation (1.1).)

2. Projective and injective resolutions

We call an R-module P *projective* if for every surjective R-module homomorphism $f : U \to V$ and R-module homomorphism $g : P \to V$, there exists an R-module homomorphism $h : P \to U$ such that $fh = g$:

$$\begin{array}{ccccc}
 & & P & & \\
 & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle g} & & \\
U & \xrightarrow{f} & V & \longrightarrow & 0
\end{array} \tag{2.1}$$

There are other equivalent definitions of projective module. For example, an R-module is projective if, and only if, it is a direct summand of a free module (i.e. $R^{\oplus I}$ for some indexing set I).

Let M be an R-module. A *projective resolution* of M is a chain complex $P_\bullet$ consisting of projective R-modules P_n $(n \geq 0)$ for which $\mathrm{H}_0(P_\bullet) \cong M$ and $\mathrm{H}_n(P_\bullet) = 0$ for all $n \neq 0$. As a consequence, $P_\bullet$ is quasi-isomorphic to the complex that is M concentrated in degree 0 and 0 elsewhere:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\varepsilon} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Another consequence of the definition is that the following sequence is exact:

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0. \tag{2.2}$$

(In some texts, (2.2) is called a projective resolution of M.) The complex (2.2) is sometimes called the *augmented complex* of $P_\bullet$. This augmented complex may be abbreviated $P_\bullet \xrightarrow{\varepsilon} M$. The complex $P_\bullet$ (without the M) is sometimes called the *truncated complex* of (2.2).

Projective resolutions of R-modules always exist: Every R-module M is a homomorphic image of a projective R-module, for example, the free module on a set of generators of M. One may use this fact to build a projective resolution as follows. Let P_0 be a projective R-module mapping surjectively to M via an R-module homomorphism ε. Let $K_1 = \mathrm{Ker}(\varepsilon)$. In turn, K_1 is a homomorphic image of some projective R-module P_1 via some R-module homomorphism $\varepsilon_1 : P_1 \to K_1$. Denote by i_1 the inclusion map $i_1 : K_1 \to P_0$ and set $d_1 = i_1\varepsilon_1$. Let $K_2 = \mathrm{Ker}(d_1)$ and continue. Visually, we have:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & P_2 & & \xrightarrow{\quad d_2 \quad} & & P_1 & & \xrightarrow{\quad d_1 \quad} & & P_0 & \xrightarrow{\varepsilon} M \longrightarrow 0 \\ & & & {\scriptstyle\varepsilon_2}\searrow & & \nearrow{\scriptstyle i_2} & & {\scriptstyle\varepsilon_1}\searrow & & \nearrow{\scriptstyle i_1} & & \\ & & & & K_2 & & & & K_1 & & & \end{array} \tag{2.3}$$

The R-module K_i is called an *ith syzygy module* of M. This module depends on some choices. However, it is unique up to an equivalence relation, as stated in Lemma 2.5 below. We will first need Schanuel's Lemma:

Lemma 2.4 (Schanuel's Lemma)**.** *Let*

$$0 \to K \to P \xrightarrow{\varepsilon} M \to 0 \quad \textit{and} \quad 0 \to K' \to P' \xrightarrow{\varepsilon'} M \to 0$$

be two short exact sequences of R-modules with P, P' projective. Then $K \oplus P' \cong K' \oplus P$.

A proof is given in class, using the pullback of ε and ε'. For definition of pullback, see the appendix.

The next lemma is a consequence of Schanuel's Lemma via a mathematical induction argument.

Lemma 2.5. *Let K_i and K_i' be two ith syzygy modules of the R-module M. There are projective R-modules P, P' such that $K_i \oplus P \cong K_i' \oplus P'$.*

A proof is given in class.

Remark 2.6. There is another way to state Lemma 2.5: Call two R-modules U and V *equivalent* if there exist projective R-modules P, P' for which $U \oplus P \cong V \oplus P'$ as R-modules. This can be shown to be an equivalence relation. The conclusion of Lemma 2.5 is that K_i and K_i' are equivalent under this equivalence relation.

The next theorem implies a relation among projective resolutions themselves.

Theorem 2.7 (Comparison Theorem). *Let $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ be chain complexes of R-modules with $M = \mathrm{H}_0(P_\bullet)$, $N = \mathrm{H}_0(Q_\bullet)$, and let $\varepsilon : P_0 \to M$ and $\varepsilon' : Q_0 \to N$ be corresponding augmentation maps. Assume that P_i is projective for each i and that the augmented complex $\cdots \to Q_1 \to Q_0 \to N \to 0$ is exact. If $f : M \to N$ is an R-module homomorphism, then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ for which $f\varepsilon = \varepsilon' f_0$, that is, the following diagram commutes:*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & Q_2 & \xrightarrow{d'_2} & Q_1 & \xrightarrow{d'_1} & Q_0 & \xrightarrow{\varepsilon'} & N & \longrightarrow & 0
\end{array}$$

The chain map $f_\bullet$ is unique up to chain homotopy.

Proof. A proof of existence of chain map $f_\bullet$ was given in class in the special case that both $P_\bullet$ and $Q_\bullet$ are projective resolutions. It can be checked that the "same" proof applies under these slightly more general hypotheses.

Proof of uniqueness up to chain homotopy: Let $g_\bullet : P_\bullet \to Q_\bullet$ be another chain map lifting $f : M \to N$, so that $f\varepsilon = \varepsilon' g_0$ and $d'_n g_n = g_{n-1} d_n$ for all $n \geq 1$. First note that as $\varepsilon'(f_0 - g_0) = f\varepsilon - f\varepsilon = 0$, we have

$$\mathrm{Im}(f_0 - g_0) \subseteq \mathrm{Ker}(\varepsilon') = \mathrm{Im}(d'_1).$$

Since P_0 is projective, there exists a map $s_0 : P_0 \to Q_1$:

$$\begin{array}{ccccc}
 & & P_0 & & \\
 & \swarrow{\scriptstyle s_0} & \downarrow{\scriptstyle f_0 - g_0} & & \\
Q_1 & \xrightarrow{d'_1} & \mathrm{Im}(d'_1) & \longrightarrow & 0
\end{array} \tag{2.8}$$

Setting $s_{-1} \equiv 0$, we now have $f_0 - g_0 = d'_1 s_0 = d'_1 s_0 + s_{-1}\varepsilon$.

Next we claim that $\mathrm{Im}(f_1 - g_1 - s_0 d_1) \subseteq \mathrm{Im}(d'_2)$. To see this, first compute the composition:

$$d'_1(f_1 - g_1 - s_0 d_1) = f_0 d_1 - g_0 d_1 - d'_1 s_0 d_1 = (f_0 - g_0) d_1 - d'_1 s_0 d_1 = d'_1 s_0 d_1 - d'_1 s_0 d_1 = 0.$$

It follows that $\mathrm{Im}(f_1 - g_1 - s_0 d_1) \subseteq \mathrm{Ker}(d'_1) = \mathrm{Im}(d'_2)$. Since P_1 is projective, there is consequently a map $s_1 : P_1 \to Q_2$:

$$\begin{array}{ccccc}
 & & P_1 & & \\
 & \swarrow{\scriptstyle s_1} & \downarrow{\scriptstyle f_1 - g_1 - s_0 d_1} & & \\
Q_2 & \xrightarrow{d'_2} & \mathrm{Im}(d'_2) & \longrightarrow & 0
\end{array} \tag{2.9}$$

So $f_1 - g_1 - s_0 d_1 = d'_2 s_1$, that is $f_1 - g_1 = s_0 d_1 + d'_2 s_1$.

Continuing in this manner, we see that for each n, there exists a map $s_n : P_n \to Q_n$ with $f_n - g_n = s_{n-1} d_n + d'_{n+1} s_n$. That is, the two chain maps $f_\bullet$ and $g_\bullet$ are chain homotopic. $\square$

As a consequence of the Comparison Theorem, if $P_\bullet, Q_\bullet$ are projective resolutions of M, N, respectively, then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ *lifting* the R-module homomorphism $f : M \to N$.

We will also work with injective resolutions. An R-module I is called *injective* if for every injective R-module homomorphism $f : V \to U$ and R-module homomorphism $g : V \to I$, there is an R-module homomorphism $h : U \to I$ for which $hf = g$:

$$\begin{array}{ccccc} & & I & & \\ & \overset{h}{\nearrow} & \uparrow g & & \\ U & \overset{f}{\longleftarrow} & V & \longleftarrow & 0 \end{array} \tag{2.10}$$

An *injective resolution* of an R-module M is a cochain complex $(I_\bullet, d_\bullet)$ consisting of injective R-modules I_n $(n \leq 0)$ for which $\mathrm{H}_0(I_\bullet) \cong M$ and $\mathrm{H}_n(I_\bullet) = 0$ for all $n \neq 0$. That is, $M \cong \mathrm{Ker}(d_0)$, and the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\iota} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots , \tag{2.11}$$

where ι is an isomorphism from M to $\mathrm{Ker}(d_0)$ followed by inclusion into I_0. The complex (2.11) is sometimes called the *augmented complex* of $I_\bullet$. It may also itself be called an injective resolution of M. The complex $I_\bullet$ is sometimes called the *truncated complex* of (2.11).

Injective resolutions of R-modules always exist: Baer's Theorem states that every R-module can be embedded into an injective R-module. (See e.g. [5] or Hungerford for a proof of Baer's Theorem.) An injective resolution can thus be built similarly (and in reverse order) to the construction of a projective resolution above. Specifically, let $L_1 = \mathrm{Coker}(\iota) = I_0/\mathrm{Im}(\iota)$. Let $\pi_0 : I_0 \to L_1$ be the quotient map. Embed L_1 into an injective module I_1 via an R-module homomorphism $\iota_1 : L_1 \to I_1$. Let $\delta_0 = \iota_1\pi_0$. Continue by letting $L_2 = \mathrm{Coker}(\iota_1) = \mathrm{Coker}(\delta_0)$, then embed L_2 into an injective module I_2, and so on.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \xrightarrow{\iota} & I_0 & & \xrightarrow{\quad\delta_0\quad} & & I_1 & & \xrightarrow{\quad\delta_1\quad} & & I_2 & \longrightarrow \cdots \\ & & & & & \underset{\pi_0}{\searrow} & & \underset{\iota_1}{\nearrow} & & \underset{\pi_1}{\searrow} & & \underset{\iota_2}{\nearrow} & & \\ & & & & & & L_1 & & & & L_2 & & & \end{array} \tag{2.12}$$

The module L_1 will be unique up to injective direct summands, due to a dual version of Schanuel's Lemma that may be checked: If $0 \to N \to I \to L \to 0$ and $0 \to N \to I' \to L' \to 0$ are exact sequences with I, I' injective, then there is an isomorphism $L \oplus I' \cong L' \oplus I$.

There is a version of Theorem 2.7 for injective resolutions. See for example [7, Comparison Theorem 2.7].

Other types of resolutions may be defined similarly, for example, flat resolutions. (See Remark 3.3 below for a definition of *flat*.)

3. Ext and Tor

In this section we will define Ext and Tor.

Ext. Let M and N be R-modules. Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M. Apply $\mathrm{Hom}_R(-,N)$ to the (truncated) complex $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$ to obtain a sequence of abelian groups (in reverse order for visual appeal):

$$0 \longrightarrow \mathrm{Hom}_R(P_0,N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1,N) \xrightarrow{d_2^*} \cdots \tag{3.1}$$

Here the abelian group homomorphism d_i^* is that induced by d_i, i.e. $d_i^*(f) = fd_i$ for all $f \in \mathrm{Hom}_R(P_{i-1},N)$ and all $i > 0$. For convenience, we define $d_0^* = 0$.

Note that $d_{i+1}^* d_i^* = 0$ since $d_i d_{i+1} = 0$, so the above sequence (3.1) is in fact a (cochain) complex of abelian groups (that is, $\mathbb{Z}$-modules). If R is commutative, it is a complex of R-modules. If R is an algebra over a field k (that is, a ring that is also a vector space with multiplication bilinear), it is a complex of k-vector spaces. Define $\mathrm{Ext}_R^n(M,N)$ to be the cohomology of this cochain complex:

$$\mathrm{Ext}_R^n(M,N) = \mathrm{H}^n(\mathrm{Hom}_R(P_\bullet,N)) = \mathrm{Ker}(d_{n+1}^*)/\,\mathrm{Im}(d_n^*)$$

for $n \geq 0$, and

$$\mathrm{Ext}_R^*(M,N) = \bigoplus_{n\geq 0} \mathrm{Ext}_R^n(M,N).$$

It can be checked that, by the Comparison Theorem (Theorem 2.7), up to isomorphism of abelian groups, $\mathrm{Ext}_R^*(M,N)$ does not depend on choice of projective resolution of M. In degree 0, we have

$$\mathrm{Ext}_R^0(M,N) \cong \mathrm{Hom}_R(M,N).$$

Note that by construction, if M is itself projective, then $\mathrm{Ext}_R^n(M,N) = 0$ for all $n > 0$.

We may alternatively define $\mathrm{Ext}_R^n(M,N)$ by first taking an injective resolution of N, instead of a projective resolution of M: Take $N \xrightarrow{\iota} I_\bullet$ to be an injective resolution of N. Apply $\mathrm{Hom}_R(M,-)$ to the (truncated) sequence $0 \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots$ to obtain:

$$0 \longrightarrow \mathrm{Hom}_R(M,I_0) \xrightarrow{(d_0)_*} \mathrm{Hom}_R(M,I_1) \xrightarrow{(d_1)_*} \cdots \tag{3.2}$$

with $(d_i)_*(f) = d_i f$ for all i and $f \in \mathrm{Hom}_R(M,I_i)$. Set $(d_{-1})_* = 0$. It can be checked that $(d_{i+1})_*(d_i)_* = 0$ for all $i \geq -1$. Thus the sequence (3.2) is a (cochain) complex of abelian groups. It can be shown that

$$\mathrm{Ext}_R^n(M,N) \cong \mathrm{H}^n(\mathrm{Hom}_R(M,I_\bullet)) = \mathrm{Ker}((d_n)_*)/\,\mathrm{Im}((d_{n-1})_*).$$

(A proof of the above isomorphism will at least be outlined in class at some point.) If N is an injective R-module, it now follows that $\mathrm{Ext}_R^n(M,N) = 0$ for all $n > 0$.

Tor. Let M be a right R-module and let N be a left R-module. Let $P_\bullet \to M$ be a (right R-module) projective resolution of M. Apply $-\otimes_R N$ to the (truncated) complex $P_\bullet$ to obtain a sequence of abelian groups (i.e. $\mathbb{Z}$-modules):

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{d_2 \otimes 1_N} P_1 \otimes_R N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_R N \longrightarrow 0.$$

Here 1_N denotes the identity map on N. (In order to reduce notational clutter, we suppress the subscript R on the tensor symbol $\otimes$, just for maps and elements, when it is clear from context that the subscript on the tensor symbol $\otimes$ should be R.) Set $d_0 = 0$. The above is a chain complex. We define $\operatorname{Tor}_n^R(M,N)$ to be its homology:

$$\operatorname{Tor}_n^R(M,N) = \mathrm{H}_n(P_\bullet \otimes_R N) = \operatorname{Ker}(d_n \otimes 1) / \operatorname{Im}(d_{n+1} \otimes 1)$$

for $n \geq 0$, and

$$\operatorname{Tor}_*^R(M,N) = \bigoplus_{n \geq 0} \operatorname{Tor}_n^R(M,N).$$

It can be checked that, by the Comparison Theorem (Theorem 2.7), $\operatorname{Tor}_n^R(M,N)$ does not depend on choice of projective resolution of M. It may be checked that

$$\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N.$$

We may alternatively define $\operatorname{Tor}_n^R(M,N)$ via a (left R-module) projective resolution of N: Let $Q_\bullet \to N$ be a projective resolution of N. Apply $M \otimes_R -$ to $Q_\bullet$ to obtain a sequence

$$\cdots \longrightarrow M \otimes_R Q_2 \xrightarrow{1_M \otimes d_2} M \otimes_R Q_1 \xrightarrow{1_M \otimes d_1} M \otimes_R Q_0 \longrightarrow 0.$$

It can be proven that $\operatorname{Tor}_n^R(M,N) \cong \mathrm{H}_n(M \otimes_R Q_\bullet)$. *(Details to be discussed in class at some point.)*

Remark 3.3. We call a left R-module F *flat* if for every short exact sequence of right R-modules $0 \to U \to V \to W \to 0$, the induced sequence of abelian groups $0 \to U \otimes_R F \to V \otimes_R F \to W \otimes_R F \to 0$ is exact. (There is a similar definition of flatness for right R-modules.) It may be checked that every projective module is flat. As a direct consequence of this definition, combined with the definition and balancing of Tor, if either M or N is flat as an R-module, then $\operatorname{Tor}_n^R(M,N) = 0$ for all $n > 0$.

4. Long exact sequences for Ext and Tor

We first state the Snake Lemma, which has many uses. Here, we will use it to construct long exact sequences for Ext and Tor.

Lemma 4.1 (Snake Lemma)**.** *Suppose there is a commuting diagram of R-modules and R-module homomorphisms with exact rows:*

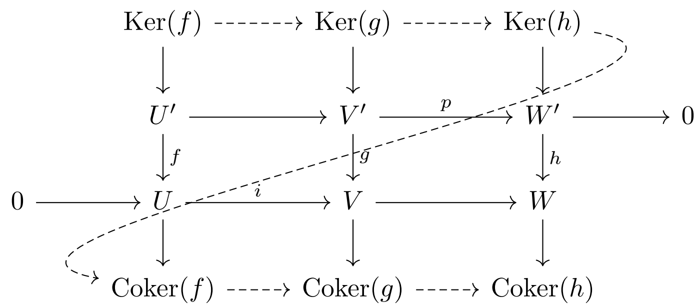
$$\begin{array}{ccccccc}
 & & U' & \longrightarrow & V' & \xrightarrow{p} & W' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & U & \xrightarrow{i} & V & \longrightarrow & W & &
\end{array}$$

Then there is an exact sequence

$$\mathrm{Ker}(f) \to \mathrm{Ker}(g) \to \mathrm{Ker}(h) \xrightarrow{\partial} \mathrm{Coker}(f) \to \mathrm{Coker}(g) \to \mathrm{Coker}(h)$$

where $\partial(w') = i^{-1}gp^{-1}(w')$ *for all* $w' \in \mathrm{Ker}(h)$.[1] *If the map* $U' \to V'$ *is injective, then the map* $\mathrm{Ker}(f) \to \mathrm{Ker}(g)$ *is injective, and if* $V \to W$ *is surjective, then* $\mathrm{Coker}(g) \to \mathrm{Coker}(h)$ *is surjective.*

This lemma is often illustrated as follows, hence the name.



One consequence of the Snake Lemma (Lemma 4.1) is the following theorem that will be used to obtain long exact sequences for Tor and Ext. We define a *short exact sequence of complexes* to be a sequence

$$0 \to U_\bullet \xrightarrow{f_\bullet} V_\bullet \xrightarrow{g_\bullet} W_\bullet \to 0$$

where $f_\bullet$, $g_\bullet$ are chain maps and f_i is injective, g_i is surjective, and $\mathrm{Im}(f_i) = \mathrm{Ker}(g_i)$ for each i.

Theorem 4.2. *Let* $0 \to U_\bullet \xrightarrow{f_\bullet} V_\bullet \xrightarrow{g_\bullet} W_\bullet \to 0$ *be a short exact sequence of complexes. There are abelian group homomorphisms* $\partial_n : \mathrm{H}_n(W_\bullet) \to \mathrm{H}_{n-1}(U_\bullet)$ *for each* n *such that*

$$\cdots \longrightarrow \mathrm{H}_{n+1}(W) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(U) \xrightarrow{\overline{f}_n} \mathrm{H}_n(V) \xrightarrow{\overline{g}_n} \mathrm{H}_n(W) \xrightarrow{\partial_n} \cdots$$

is an exact sequence. (Here, $\overline{f}_n, \overline{g}_n$ *denote the maps induced by* f_n, g_n*.)*

We call the homomorphisms ∂_n in the theorem *connecting homomorphisms.*

The next lemma will also be used in constructing long exact sequences for Ext and Tor. It is called the Horseshoe Lemma due to the shape of the given diagram.

Lemma 4.3 (Horseshoe Lemma)**.** *Supppose there is a short exact sequence of* R*-modules and* R*-module homomorphisms* $0 \to U' \to U \to U'' \to 0$*. Let* $P'_\bullet \to U'$,

[1]Caution: This is abuse of notation. The R-module homomorphisms p, i are not assumed to be isomorphisms. Instead, by $p^{-1}(w')$ we mean any element in the inverse image of w' under p, and similarly for i^{-1}, which is then followed by canonical projection to $\mathrm{Coker}(f)$. It can be checked that this element in $\mathrm{Coker}(f)$ does not depend on these choices.

$P''_\bullet \to U''$ be projective resolutions of U' and U'' as in the diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
\cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & U' & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & U & & \\
 & & & & & & \downarrow & & \\
\cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & U'' & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

For each n, let $P_n = P'_n \oplus P''_n$. Then there are differentials d_i for which

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \to U \to 0$$

is a projective resolution of U. Furthermore, the right column lifts to an exact sequence of complexes $0 \to P'_\bullet \xrightarrow{\iota_\bullet} P_\bullet \xrightarrow{\pi_\bullet} P''_\bullet \to 0$ for which $\iota_\bullet, \pi_\bullet$ are the canonical inclusion and projection maps.

Now we may use Theorem 4.2 and the Horseshoe Lemma (Lemma 4.3) to obtain the following four long exact sequences. We provide proofs of two of these; the others are similar.

Theorem 4.4 (First long exact sequence for Ext)**.** *Suppose U is an R-module and $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of R-modules. Then there is an exact sequence*

$$0 \to \operatorname{Hom}_R(U, V') \longrightarrow \operatorname{Hom}_R(U, V) \longrightarrow \operatorname{Hom}_R(U, V'') \longrightarrow$$

$$\operatorname{Ext}^1_R(U, V') \longrightarrow \operatorname{Ext}^1_R(U, V) \longrightarrow \operatorname{Ext}^1_R(U, V'') \longrightarrow \operatorname{Ext}^2_R(U, V') \cdots$$

Proof. Let $P_\bullet$ be a projective resolution of U. By left exactness of $\operatorname{Hom}_R(P_n, -)$, for each n, there are exact sequences

$$0 \to \operatorname{Hom}_R(P_n, V') \to \operatorname{Hom}_R(P_n, V) \to \operatorname{Hom}_R(P_n, V'').$$

We claim that projectivity of P_n also implies surjectivity of the rightmost map above. To see this, consider the diagram

$$\begin{array}{ccccccccc}
 & & & & & & P_n & & \\
 & & & & & \swarrow & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0
\end{array}$$

For any R-module homomorphism from P_n to V'' as indicated above, projectivity of P_n implies existence of an R-module homomorphism as indicated by the dashed arrow. Thus the map $\mathrm{Hom}_R(P_n, V) \to \mathrm{Hom}_R(P_n, V'')$ is surjective and we indeed have a short exact sequence

$$0 \to \mathrm{Hom}_R(P_n, V') \to \mathrm{Hom}_R(P_n, V) \to \mathrm{Hom}_R(P_n, V'') \to 0$$

for each n. Now it can be checked that the maps in these short exact sequences commute with maps induced by differentials (since these are composing with differentials on the right, while the above maps are given by composing on the left). Therefore, there is a short exact sequence of complexes

$$0 \to \mathrm{Hom}_R(P_\bullet, V') \to \mathrm{Hom}_R(P_\bullet, V) \to \mathrm{Hom}_R(P_\bullet, V'') \to 0.$$

Applying Theorem 4.2 to this short exact sequence of complexes, there is a long exact sequence as claimed. □

Theorem 4.5 (Second long exact sequence for Ext)**.** *Suppose V is an R-module and $0 \to U' \to U \to U'' \to 0$ is a short exact sequence of R-modules. Then there is an exact sequence*

$$0 \to \mathrm{Hom}_R(U'', V) \longrightarrow \mathrm{Hom}_R(U, V) \longrightarrow \mathrm{Hom}_R(U', V) \longrightarrow$$

$$\mathrm{Ext}^1_R(U'', V) \longrightarrow \mathrm{Ext}^1_R(U, V) \longrightarrow \mathrm{Ext}^1_R(U', V) \longrightarrow \mathrm{Ext}^2_R(U'', V) \cdots$$

Theorem 4.6 (First long exact sequence for Tor)**.** *Suppose V is a left R-module and $0 \to U' \to U \to U'' \to 0$ is an exact sequence of right R-modules. Then there is an exact sequence*

$$\cdots \longrightarrow \mathrm{Tor}^R_2(U'', V) \longrightarrow \mathrm{Tor}^R_1(U', V) \longrightarrow \mathrm{Tor}^R_1(U, V) \longrightarrow$$

$$\mathrm{Tor}^R_1(U'', V) \longrightarrow U' \otimes_R V \longrightarrow U \otimes_R V \longrightarrow U'' \otimes_R V \longrightarrow 0.$$

Proof. Apply the Horseshoe Lemma (Lemma 4.3) to $0 \to U' \to U \to U'' \to 0$ to obtain an exact sequence of projective resolutions

$$0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0,$$

noting that we may take $P_n = P'_n \oplus P''_n$ for each n. Tensor this sequence on the right with B. Since $P_n = P'_n \oplus P''_n$ for each n, and tensor product distributes over direct sum, we have an exact sequence of complexes,

$$0 \to P'_\bullet \otimes_R B \to P_\bullet \otimes_R B \to P''_\bullet \otimes_R B \to 0.$$

By Theorem 4.2, this induces the claimed long exact sequence of homology. □

Theorem 4.7 (Second long exact sequence for Tor)**.** *Suppose U is a right R-module and $0 \to V' \to V \to V'' \to 0$ is an exact sequence of left R-modules. Then there is an exact sequence*

$$\cdots \longrightarrow \mathrm{Tor}^R_2(U, V'') \longrightarrow \mathrm{Tor}^R_1(U, V') \longrightarrow \mathrm{Tor}^R_1(U, V) \longrightarrow$$

$$\mathrm{Tor}^R_1(U, V'') \longrightarrow U \otimes_R V' \longrightarrow U \otimes_R V \longrightarrow U \otimes_R V'' \longrightarrow 0.$$

Remark 4.8. Recall Remark 3.3 on flat modules. As a consequence of the above Theorem 4.6 (respectively, Theorem 4.7), if V is a left R-module and $\operatorname{Tor}_1^R(M,V) = 0$ for all right R-modules M, then V is flat (respectively, if U is a right R-module and $\operatorname{Tor}_1^R(U,M) = 0$ for all left R-modules M, then U is flat).

5. Homological dimensions and Hilbert's Syzygy Theorem

In this section, we will define the homological dimensions termed projective dimension and global dimension, focusing first on the special cases of rings and modules for which these are either 0 or 1. We will then develop some preliminary results leading to a proof of Hilbert's Syzygy Theorem that gives the global dimension of a polynomial ring.

We will begin with semisimple rings and modules. An R-module M is *semisimple* if it is isomorphic to a direct sum of (possibly infinitely many) simple R-modules. (Recall a *simple* R-module is one having no proper nonzero R-submodules.) The ring R is called *semisimple* if the R-module R itself (under either left or right multiplication) is a semisimple R-module.

A classification of semsimple rings is given by the following theorem.

Theorem 5.1 (Wedderburn-Artin Theorem)**.** *Every semisimple ring is isomorphic to a finite direct product of matrix rings over division rings.*

The following proposition gives several equivalent conditions for a ring to be semisimple.

Proposition 5.2. *The following are equivalent for a ring R.*

(i) *R is semisimple.*
(ii) *Every R-module is semisimple.*
(iii) *Every R-module is projective.*
(iv) *Every R-module is injective.*

We will turn to nonsemisimple rings next, and start by proving a theorem that is sometimes termed *dimension shifting*. It gives an isomorphism between Ext groups in different homological degrees by replacing a module with one of its syzygy modules (thus shifting degree, or dimension, of the Ext group).

For each R-module M and $i \geq 1$, denote by $\Omega^i M$ an ith syzygy module of M. (The statements of theorems to follow will not depend on the choice of $\Omega^i M$.)

Theorem 5.3 (Dimension shifting)**.** *Let M and N be R-modules. There is an isomorphism of abelian groups for all $n \geq 2$,*

$$\operatorname{Ext}_R^n(M,N) \quad \cong \quad \operatorname{Ext}_R^1(\Omega^{n-1}M, N).$$

Proof. This is by construction of diagram (2.3) and definition of Ext. Alternatively, one may apply the second long exact sequence for Ext to the short exact sequence $0 \to \Omega^1 M \to P_0 \to M \to 0$ and N to obtain isomorphisms $\operatorname{Ext}_R^n(M,N) \cong \operatorname{Ext}_R^{n-1}(\Omega^1 M, N)$. Then note $\Omega^i(\Omega^1 M)$ is isomorphic to $\Omega^{i+1}M$ up to projective direct summands, and use induction. □

Let V be a left R-module. The *projective dimension* $\operatorname{pd}_R(V)$ is the smallest integer n for which there exists a projective resolution of V of the form

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to V \to 0.$$

If no such n exists, we write $\mathrm{pd}_R(V) = \infty$. By its definition, we see that the projective dimension $\mathrm{pd}_R(M)$ of M is also the smallest integer n for which some nth syzygy module $\Omega^n M$ of M is projective. (Note this does not depend on choice of syzygy module since any two are isomorphic up to projective direct summands as seen earlier.)

The *left global dimension* of R is

$$\mathrm{gldim}(R) = \sup\{\mathrm{pd}_R(V) \mid V \text{ is a left } R\text{-module}\}.$$

(This is also sometimes denoted $\mathrm{lgldim}(R)$ to distinguish it from the similarly defined right global dimension, which can be different.) Note that R is semisimple if, and only if, $\mathrm{gldim}(R) = 0$.

A ring R is called (left) *hereditary* if every left R-submodule of every free left R-module is projective.

Proposition 5.4. $\mathrm{gldim}(R) = 1$ *if, and only if, R is hereditary and not semisimple.*

By dimension shifting and the definition of projective dimension, we find the following.

Lemma 5.5. *Let A be a left R-module and d be a nonnegative integer. The following are equivalent:*

(i) $\mathrm{pd}_R(A) \le d$.
(ii) $\mathrm{Ext}^n_R(A, B) = 0$ *for all $n > d$ and all left R-modules B.*
(iii) $\mathrm{Ext}^{d+1}_R(A, B) = 0$ *for all left R-modules B.*

As a consequence,

$$\mathrm{pd}_R(M) = \sup\{j \mid \mathrm{Ext}^j_R(M, N) \ne 0 \text{ for some } R\text{-module } N\}. \tag{5.6}$$

The next lemma gives a comparison of projective dimensions among modules in a short exact sequence.

Lemma 5.7. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of left R-modules. Then*

$$\mathrm{pd}_R(B) \le \max\{\mathrm{pd}_R(A), \mathrm{pd}_R(C)\}.$$

If the inequality is strict, then $\mathrm{pd}_R(C) = \mathrm{pd}_R(A) + 1$.

The next two theorems and corollary compare projective dimensions of modules for two rings related to each other in the specified ways.

Theorem 5.8 (First Change of Rings Theorem)**.** *Let x be a central nonzerodivisor in R and let $\overline{R} = R/(x)$. If A is a nonzero left $\overline{R}$-module with $\mathrm{pd}_{\overline{R}}(A)$ finite, then*

$$\mathrm{pd}_R(A) = \mathrm{pd}_{\overline{R}}(A) + 1.$$

Proof. Note that as an R-module, A is not projective, since $xA = 0$. In particular, $\overline{R}$, considered as an R-module, is not projective.

We will prove the statement by induction on $\mathrm{pd}_{\overline{R}}(A)$.

Suppose first that $\mathrm{pd}_{\overline{R}}(A) = 0$, that is, A is projective as an $\overline{R}$-module. Then A is a direct summand of a free $\overline{R}$-module, which is itself a direct sum of copies of $\overline{R}$. We consider the simplest case $A = \overline{R}$, and recall that it has projective resolution as an R-module:

$$0 \longrightarrow R \xrightarrow{x\cdot} R \xrightarrow{\pi} \overline{R} \longrightarrow 0$$

where π is the quotient map. Thus R is a first syzygy module of $\overline{R}$, considered as an R-module, and it follows that $\mathrm{pd}_R(\overline{R}) = 1$. The same is true of any direct sum of copies of $\overline{R}$ by (5.6) and properties of Ext.[2] Again, by similar reasoning, the same will be true of any direct summand A of a free $\overline{R}$-module, that is, $\mathrm{pd}_R(A) = 1$.

Now assume that $\mathrm{pd}_{\overline{R}}(A) > 0$ and that for all $\overline{R}$-modules B with $\mathrm{pd}_{\overline{R}}(B) < \mathrm{pd}_{\overline{R}}(A)$, we have $\mathrm{pd}_R(B) = \mathrm{pd}_{\overline{R}}(B) + 1$. Let

$$0 \to K \to P \to A \to 0 \tag{5.9}$$

be a short exact sequence of $\overline{R}$-modules for which P is projective as an $\overline{R}$-module. Since we have assumed that $\mathrm{pd}_{\overline{R}}(A) > 0$, A is not projective as an $\overline{R}$-module, and it follows that

$$\mathrm{pd}_{\overline{R}}(A) = \mathrm{pd}_{\overline{R}}(K) + 1.$$

By the induction hypothesis, $\mathrm{pd}_R(K) = \mathrm{pd}_{\overline{R}}(K) + 1$. From the above equation, it also now follows that $\mathrm{pd}_R(K) = \mathrm{pd}_{\overline{R}}(A)$.

We wish to prove that $\mathrm{pd}_R(A) = \mathrm{pd}_{\overline{R}}(A) + 1$, which by the above equations is equivalent to proving that $\mathrm{pd}_R(A) = \mathrm{pd}_R(K) + 1$. Suppose this is not the case. Consider the sequence (5.9) as a short exact sequence of R-modules. Since we have assumed that $\mathrm{pd}_R(A) \neq \mathrm{pd}_R(K) + 1$, by Lemma 5.7,

$$\mathrm{pd}_R(P) = \max\{\mathrm{pd}_R(K), \mathrm{pd}_R(A)\}. \tag{5.10}$$

By the first part of the proof, we know that $\mathrm{pd}_R(P) = 1$. Also, since A and K are $\overline{R}$-modules, neither can be projective as an R-module, and so this now forces $\mathrm{pd}_R(A) = 1 = \mathrm{pd}_R(K)$. Since $\mathrm{pd}_{\overline{R}}(A) = \mathrm{pd}_R(K)$, it also follows that $\mathrm{pd}_{\overline{R}}(A) = 1$. We will show that this is not possible, that is, it is not possible that both $\mathrm{pd}_R(A) = 1$ and $\mathrm{pd}_{\overline{R}}(A) = 1$.

Let F be a free R-module mapping onto A, with kernel L:

$$0 \to L \to F \to A \to 0.$$

Since $\mathrm{pd}_R(A) = 1$, we have $\mathrm{pd}_R(L) = 0$, that is, L is projective as an R-module. Now apply Theorem 4.7 (second long exact sequence for Tor) to this sequence and to $\overline{R}$ considered as a right R-module to obtain an exact sequence of abelian groups (note the 0 term on the left is due to F being a free R-module):

$$0 \to \mathrm{Tor}_1^R(\overline{R}, A) \to \overline{R} \otimes_R L \to \overline{R} \otimes_R F \to \overline{R} \otimes_R A \to 0$$

Note that the abelian groups $\overline{R} \otimes_R L$, $\overline{R} \otimes_R F$, and $\overline{R} \otimes_R A$ in the above sequence are all in fact $\overline{R}$-modules (with action by multiplication on the leftmost tensor factor) and the maps between them are in fact $\overline{R}$-module homomorphisms. Therefore the term $\mathrm{Tor}_1^R(\overline{R}, A)$ is also an $\overline{R}$-module (being a homology group for a complex of $\overline{R}$-modules) and the above is an exact sequence of $\overline{R}$-modules. Now, since $xA = 0$, we have $\overline{R} \otimes_R A \cong A/xA \cong A$ and by Homework Assignment 1, #3, $\mathrm{Tor}_1^R(\overline{R}, A) \cong \{a \in A \mid xa = 0\} \cong A$. So the above exact sequence is equivalent to:

$$0 \to A \to \overline{R} \otimes_R L \to \overline{R} \otimes_R F \to A \to 0$$

---

[2] See e.g. [6, Proposition 7.21] for the isomorphism

$$\mathrm{Ext}_R^n\left(\bigoplus_{i \in I} A_i, B\right) \cong \prod_{i \in I} \mathrm{Ext}_R^n(A_i, B).$$

Note that since L and F are projective as R-modules (i.e. direct summmands of free R-modules), $\overline{R}\otimes_R L$ and $\overline{R}\otimes_R F$ are projective as $\overline{R}$-modules. Since $\mathrm{pd}_{\overline{R}}(A)=1$, the kernel K' of the map $\overline{R}\otimes_R F\to A$ is projective as an $\overline{R}$-module. Therefore it is a direct summand of $\overline{R}\otimes_R L$, with the leftmost copy of A as its complement (due to the exactness of the sequence and the projectivity of K'). We have shown that A is projective as an $\overline{R}$-module, contradicting our assumption. Therefore it is not true that both $\mathrm{pd}_R(A)=1$ and $\mathrm{pd}_{\overline{R}}(A)=1$. Consequently, $\mathrm{pd}_R(A)=\mathrm{pd}_{\overline{R}}(A)+1$, as desired. □

Theorem 5.11 (Second Change of Rings Theorem)**.** *Let x be a central nonzerodivisor in a ring R, and let $\overline{R}=R/(x)$. If A is a left R-module and* $\mathrm{Tor}_1^R(\overline{R},A)=0$, *then*

$$\mathrm{pd}_R(A)\geq \mathrm{pd}_{\overline{R}}(A/xA).$$

Corollary 5.12. *Let S be a ring, $R=S[x]$, and M a left S-module. Then*

$$\mathrm{pd}_R(R\otimes_S M)=\mathrm{pd}_S(M).$$

Theorem 5.13. *Let S be a ring. Then*

$$\mathrm{gldim}\,(S[x_1,\dots,x_n])=\mathrm{gldim}(S)+n.$$

Remark 5.14. There are right module versions of Theorem 5.8, Theorem 5.11, Corollary 5.12, and Theorem 5.13, proved similarly.

Hilbert's Syzygy Theorem is now an immediate consequence of Theorem 5.13.

Theorem 5.15 (Hilbert's Syzygy Theorem)**.** *Let k be a field. Then*

$$\mathrm{gldim}\,(k[x_1,\dots,x_n])=n.$$

Thus the nth syzygy module of every $k[x_1,\dots,x_n]$-module is projective.

Remark 5.16. For another class of examples, it can be shown that $\mathrm{gldim}(k[[x_1,\dots,x_n]])=n$, as a consequence of a more general statement [1]. There are also analogs for noncommutative versions of polynomial rings (see e.g. [3, Section 7.5]).

6. Categories, functors, derived functors

Everything we have done so far in the course, other than results for specific types of rings and modules, can be done more generally for categories having some good properties (specifically, additive or abelian categories). We will start by summarizing the needed ideas from category theory. Then we will define derived functors, showing that Ext and Tor are specific instances of derived functors.

Categories and functors. Recall that a *category* $\mathcal{C}$ consists of a collection of *objects* $\mathrm{Obj}(\mathcal{C})$ and sets of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(A,B)$ for all objects A,B of $\mathcal{C}$, satisfying the following: For each object A in $\mathcal{C}$, there is an *identity morphism* $1_A\in\mathrm{Hom}_{\mathcal{C}}(A,A)$. There is a binary operation, called *composition*, $\circ:\mathrm{Hom}_{\mathcal{C}}(A,B)\times\mathrm{Hom}_{\mathcal{C}}(B,C)\to\mathrm{Hom}_{\mathcal{C}}(A,C)$ for all objects A,B,C of $\mathcal{C}$, for which

$$(hg)f=h(gf)\quad\text{and}\quad 1_Bf=f1_A$$

for all $f\in\mathrm{Hom}_{\mathcal{C}}(A,B)$, $g\in\mathrm{Hom}_{\mathcal{C}}(B,C)$, $h\in\mathrm{Hom}_{\mathcal{C}}(C,D)$ and all objects A,B,C,D of $\mathcal{C}$. (In the above, for simplicity, we have written gf in place of $g\circ f$ to denote composition of morphisms.)

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ is called an *isomorphism* if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B,A)$ for which $gf = 1_A$ and $fg = 1_B$.

Examples of categories include those of modules for a ring R: Let R-Mod (respectively, R-mod) denote the category of all left R-modules (respectively, all finitely generated left R-modules). The objects are the modules, the morphisms are module homomorphisms (including the identity homomorphism for each module), and composition is function composition. Similarly, let Mod-R (respectively, mod-R) denote the category of right R-modules (respectively, all finitely generated right R-modules). It is common to simplify the notation $\mathrm{Hom}_{R\text{-Mod}}$ (respectively, $\mathrm{Hom}_{R\text{-mod}}$, $\mathrm{Hom}_{\text{Mod-}R}$, $\mathrm{Hom}_{\text{mod-}R}$) and write just Hom_R in all these cases; it should always be clear from context which is meant.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is an assignment of an object $F(A)$ of $\mathcal{D}$ to each object A of $\mathcal{C}$, and a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ for each pair of objects A, B of $\mathcal{C}$ such that the following holds: $F(1_A) = 1_{F(A)}$ for all A and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms f, g that it is possible to compose. One example of a functor is the identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ given by $1_{\mathcal{C}}(A) = A$ and $1_{\mathcal{C}}(f) = f$ for all objects A and morphisms f of $\mathcal{C}$.

To be more precise, the above in fact defines a *covariant functor*. Similarly, a *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is an assignment to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(B), F(A))$ for which $F(1_A) = 1_{F(A)}$ and $F(g \circ f) = F(f) \circ F(g)$ for all morphisms f, g that can be composed.

Natural transformations. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \to G$ is an assignment of a morphism $\eta_A : F(A) \to G(A)$ to each object A of $\mathcal{C}$ for which the following holds: $G(f) \circ \eta_A = \eta_B \circ F(f)$ for all objects A, B of $\mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$. That is, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

We say that η is a *natural isomorphism* if η_A is an isomorphism for each object A. In this case, we write $F \cong G$.

Remark 6.1. Often in the literature, the term "natural" is used in this technical sense.

We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to D$, $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $FG \cong 1_{\mathcal{D}}$ and $GF \cong 1_{\mathcal{C}}$.

Example 6.2. Let k be a field and $M_2(k)$ be the ring of 2×2-matrices over k. Each of these two rings is semisimple, with a unique simple module up to isomorphism. This leads to a category equivalence between their module categories (sending the unique simple object of one to the unique simple object of the other). When two rings have equivalent categories of modules via additive functors (see definition of additive functor below) as in this example, the rings are called *Morita equivalent*.

Adjoint functors. Next we define adjoint functors. Let $\mathcal{C}$, $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F is *left adjoint* to G, and that G is *right adjoint* to F if there exist natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

for each object X of $\mathcal{C}$ and Y of $\mathcal{D}$.

Standard examples of adjoint functors are given by induction and coinduction of modules: Let R be a ring with a subring $S \subseteq R$. Let N be an S-module. The corresponding *induced* (also called *tensor induced*) R-module is $R \otimes_S N$ as an abelian group, with R-module structure given by multiplication on the left tensor factor R. The corresponding *coinduced* R-module is $\mathrm{Hom}_S(R, N)$ as an abelian group with R-module structure given by

$$(r \cdot f)(x) = f(xr)$$

for all $r, x \in R$ and $f \in \mathrm{Hom}_S(R, N)$. The following lemma states that these are left and right adjoint functors of restriction of modules from R to S:

Lemma 6.3 (Nakayama relations)**.** *Let R be a ring with a subring $S \subseteq R$. Let M be an R-module and N an S-module. There are isomorphisms of abelian groups:*

$$\begin{aligned} \mathrm{Hom}_S(N, M) &\cong \mathrm{Hom}_R(R \otimes_S N, M), \\ \mathrm{Hom}_S(M, N) &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(R, N)). \end{aligned}$$

That is, induction is left adjoint to restriction from R to S, and coinduction is right adjoint to restriction from R to S.

The above Lemma 6.3 may be used to prove:

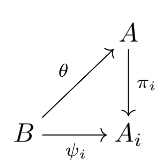
Lemma 6.4 (Eckmann-Shapiro Lemma)**.** *Let R be a ring with a subring $S \subseteq R$ for which R is projective as a right S-module under multiplication. Let M be an R-module and N an S-module. Then there are isomorphisms of abelian groups:*

$$\begin{aligned} \mathrm{Ext}^n_S(N, M) &\cong \mathrm{Ext}^n_R(R \otimes_S N, M), \\ \mathrm{Ext}^n_S(M, N) &\cong \mathrm{Ext}^n_R(M, \mathrm{Hom}_S(R, N)). \end{aligned}$$

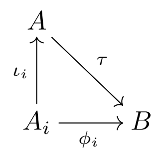
Additive and abelian categories. We define additive and abelian categories next; these are categories that have enough structure for homological algebra. Categories of modules for rings will be examples. There are some other definitions we need to make first.

A *zero object* of a category $\mathcal{C}$ is an object Z for which $|\mathrm{Hom}_{\mathcal{C}}(Z, X)| = 1$ and $|\mathrm{Hom}_{\mathcal{C}}(X, Z)| = 1$ for all objects X of $\mathcal{C}$ (we also say that Z is both an *initial* and a *terminal* object). Such a zero object Z is often denoted 0.

We recall next the notions of product and coproduct: Let $\{A_i\}_{i \in I}$ be a set of objects A_i of $\mathcal{C}$ indexed by some set I. A *product* $\prod_{i \in I} A_i$ is an object A and a collection of morphisms $\pi_i \in \mathrm{Hom}_{\mathcal{C}}(A, A_i)$ for all $i \in I$ for which the following universal property holds: If B is an object of $\mathcal{C}$ and $\psi_i \in \mathrm{Hom}_{\mathcal{C}}(B, A_i)$ for all $i \in I$, then there exists a unique morphism $\theta \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that the following diagram commutes for all $i \in I$:



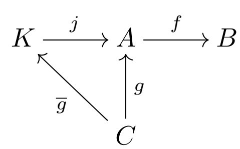
A *coproduct* $\coprod_{i\in I} A_i$ is an object A and a collection of morphisms $\iota_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, A)$ for which the following universal property holds: If B is an object of $\mathcal{C}$ and $\phi_i \in \mathrm{Hom}_{\mathcal{C}}(A_i, B)$ for all $i \in I$, then there is a unique $\tau \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ such that the following diagram commutes for all $i \in I$:



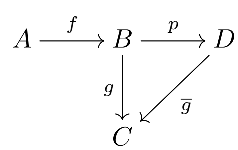
As examples, in categories of modules, product is direct product and coproduct is direct sum.

A category $\mathcal{C}$ is called *additive* if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group (i.e. $\mathbb{Z}$-module) for each pair of objects A, B in $\mathcal{C}$, composition of morphisms is $\mathbb{Z}$-bilinear, and $\mathcal{C}$ has a zero object, finite products and coproducts. Let $\mathcal{C}$ and $\mathcal{D}$ be two additive categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is *additive* if F induces homomorphisms of abelian groups $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ for all objects A, B of $\mathcal{C}$. By [2, Proposition II.9.5], this is equivalent to preserving products of two objects.

We next define kernels and cokernels for morphisms in additive categories, by way of universal properties. A slight difference in comparison with modules for a ring is that these are defined not just as objects but as pairs of objects and morphisms: Let $\mathcal{C}$ be an additive category and consider a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ for objects A, B of $\mathcal{C}$. A *kernel* of f is an object K in $\mathcal{C}$ and a morphism $j \in \mathrm{Hom}_{\mathcal{C}}(K, A)$ such that $fj = 0$, and whenever C is an object and $g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ satisfies $fg = 0$, there is a unique $\overline{g} \in \mathrm{Hom}_{\mathcal{C}}(C, K)$ such that $j\overline{g} = g$. That is, the following diagram commutes:



A *cokernel* of f is an object D in $\mathcal{C}$ and a morphism $p \in \mathrm{Hom}_{\mathcal{C}}(B, D)$ such that $pf = 0$, and whenever C is an object and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ satisfies $gf = 0$, there is a unique $\overline{g} \in \mathrm{Hom}_{\mathcal{C}}(D, C)$ such that $\overline{g}p = g$. That is, the following diagram commutes:



It may be checked that kernels and cokernels are unique up to isomorphism (of the objects), and also that they can be realized as pullbacks and pushouts, respectively. When all is clear from context, sometimes by kernel or cokernel, it is common to refer to just the object, and other times to just the morphism.

We define monomorphism and epimorphism next. Let $\mathcal{C}$ be a category. Let A, B be objects of $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. We call f a *monomorphism* if whenever C is an object of $\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(C, A)$, if $fg = fh$ then $g = h$. We call f an *epimorphism* if whenever C is an object of $\mathcal{C}$ and $g, h \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, if $gf = hg$ then $g = h$.

A category $\mathcal{C}$ is called *abelian* if it is additive, every morphism has both a kernel and a cokernel, every monomorphism is a kernel, and every epimorphism is a cokernel. Examples include categories of R-modules for a ring R.

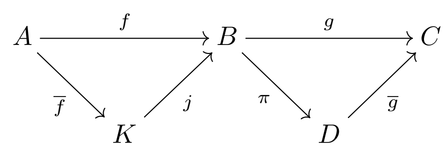
Remark 6.5. As a consequence of the Freyd-Mitchell Embedding Theorem, your every day abelian category will in fact be equivalent to a subcategory of a category of modules for some ring.

Exact sequences and homology. Next we define exact sequences and homology for an abelian category $\mathcal{C}$: A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{6.6}$$

of objects A, B, C and morphisms f, g in $\mathcal{C}$ is called *exact* if f is a kernel of g and g is a cokernel of f. If it is a complex (not necessarily exact), that is, $gf = 0$, we define the *homology* of the complex (6.6) at B: Let $j : K \to B$ be a kernel for g, and $\pi : B \to D$ a cokernel for f. Since $gf = 0$, there exist morphisms $\overline{f} : A \to K$ and $\overline{g} : D \to C$ for which each triangle in the diagram below commutes.

(6.7)



Let $H = \text{Coker}(\overline{f})$. It turns out that $H \cong \text{Ker}(\overline{g})$; for a detailed proof, see [4, Theorem 7.45]. We define to be the homology of (6.6) at B to be H.

We may define projective and injective objects of $\mathcal{C}$ via diagrams (2.1) and (2.10) just as in the case of a category of modules over a ring. We may similarly define projective and injective resolutions, however they do not always exist. When they do exist, homological constructions can be carried out in $\mathcal{C}$ just as for modules in the previous sections.

Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is called *left exact* (respectively, *right exact*) if for every exact sequence $0 \to A \to B \to C$ (respectively, $A \to B \to C \to 0$), the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact (respectively, $F(A) \to F(B) \to F(C) \to 0$ is exact). For example, letting $\mathcal{C}$ be the category of modules for a ring R, $\mathcal{D}$ the category of abelian groups, and M an R-module, the functor $\text{Hom}_R(M, -)$ is left exact. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is called *left exact* (respectively, *right exact*) if for every exact sequence $A \to B \to C \to 0$ (respectively, $0 \to A \to B \to C$), the sequence

$$0 \to F(C) \to F(B) \to F(A)$$

is exact (respectively, $F(C) \to F(B) \to F(A) \to 0$ is exact). For example, $\text{Hom}_R(-, M)$ is left exact. In either case, we say that F is *exact* if it is both left and right exact.

Derived functors. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $F : \mathcal{C} \to \mathcal{D}$ an additive (covariant) functor. Assume $\mathcal{C}$ *has enough projectives*, that is, assume that for each object A of $\mathcal{C}$, there is an epimorphism from a projective object in $\mathcal{C}$ to A. For each object A in $\mathcal{C}$, choose a projective resolution $(P_\bullet, d_\bullet)$ of A, which exists since $\mathcal{C}$ has enough projectives. Apply the functor F to obtain the complex $F(P_\bullet)$:

$$F(P_\bullet): \qquad \cdots \longrightarrow F(P_2) \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \longrightarrow 0.$$

The *left derived functor* of F is defined to be $L_\bullet F$ where $L_n F(A) = \mathrm{H}_n(F(P_\bullet))$. In the case that F is right exact, it follows from the definition that $L_0 F(A) \cong F(A)$. (The adjective "left" here indicates the objects are on the left with 0 at the end.)

We now show that Tor is an example of left derived functor: Let R be a ring, $\mathcal{C} = \text{Mod-}R$, $\mathcal{D} = \mathbb{Z}\text{-Mod}$, B an object of R-Mod, and F the functor $-\otimes_R B$. Then

$$L_n F(A) \cong \mathrm{Tor}_n^R(A, B).$$

Next assume $\mathcal{D}$ *has enough injectives*, that is, assume that for each object A of $\mathcal{C}$, there is a monomorphism from A to an injective object. For each object B in $\mathcal{C}$, choose an injective resolution $(I_\bullet, d_\bullet)$ of B. Apply the functor F to obtain the complex $F(I_\bullet)$:

$$F(I_\bullet): \qquad 0 \longrightarrow F(I_0) \longrightarrow F(I_1) \longrightarrow F(I_2) \longrightarrow \cdots$$

The *right derived functor* of F is defined to be $R^\bullet F$ where $R^n F(B) = \mathrm{H}^n(F(I_\bullet))$. If F is left exact, it follows from the definition that $R^0 F(B) \cong F(B)$. (The adjective "right" here indicates the objects are on the right with 0 at the beginning.)

As an example, let R be a ring, $\mathcal{C} = R\text{-Mod}$, $\mathcal{D} = \mathbb{Z}\text{-Mod}$, A an object of $\mathcal{C}$, and $F(B) = \mathrm{Hom}_R(A, B)$. Then

$$R^n F(B) \cong \mathrm{Ext}_R^n(A, B).$$

Similarly we may define derived functors of contravariant functors, using a projective resolution for a right derived functor, and an injective resolution for a left derived functor.

7. Double complexes

A *double complex* (or *bicomplex*) in an abelian category $\mathcal{A}$ (e.g. a category of modules for a ring R) is a set $B = \{B_{i,j}\}_{i,j\in\mathbb{Z}}$ of objects $B_{i,j}$ of $\mathcal{A}$ with maps

$$d_{i,j}^h : B_{i,j} \to B_{i-1,j} \quad \text{and} \quad d_{i,j}^v : B_{i,j} \to B_{i,j-1}$$

(the *horizontal* and *vertical* differentials) satisfying

$$d^h d^h = 0, \quad d^v d^v = 0, \quad \text{and} \quad d^v d^h + d^h d^v = 0.$$

The double complex B is *bounded* if for each n, only finitely many $B_{i,j}$ with $i+j=n$ are nonzero. The *total complex* of the double complex B is the complex $\mathrm{Tot}(B)$ with

$$\mathrm{Tot}(B)_n = \bigoplus_{i+j=n} B_{i,j}$$

for each n, and differential $d = d^h + d^v$. (More precisely, we set $\mathrm{Tot}^\oplus(B)_n = \oplus_{i+j=n} B_{i,j}$ and $\mathrm{Tot}^\Pi(B)_n = \prod_{i+j=n} B_{ij}$. When the double complex B is bounded, there is no difference between these two.) The *homology* of the double complex B is by definition the homology of $\mathrm{Tot}(B)$, and is often written simply as $\mathrm{H}_\bullet(B)$ rather than $\mathrm{H}_\bullet(\mathrm{Tot}(B))$.

Some important double complexes arise as tensor product complexes and Hom complexes of R-modules. We turn to these next.

Tensor product complexes. We will fix a standard sign convention in these notes, sometimes called the *Koszul sign convention*: Let V, V' be right R-modules and let W, W' be left R-modules, and suppose that each is graded by $\mathbb{N}$ or $\mathbb{Z}$, that is, each is a direct sum of submodules indexed by $\mathbb{N}$ or $\mathbb{Z}$:

$$V = \bigoplus_{n\in\mathbb{Z}} V_n,$$

and similarly for V', W, W'. An element of V_n for some n is called *homogeneous of degree* n. Let $f : V \to V'$ and $g : W \to W'$ be R-module homomorphisms that are graded in the sense that there is an integer m for which

$$f(V_n) \subseteq V_{n+m}$$

for all n, and similarly for g. In this case, write $|f| = m$, and call it the *degree* of f. The function $f \otimes_R g$ on $V \otimes_R W$ is defined by

$$(f \otimes_R g)(v \otimes_R w) = (-1)^{|g||v|} f(v) \otimes_R g(w) \tag{7.1}$$

for all homogeneous elements $v \in V$, $w \in W$.

Let $(C_\bullet, d^C_\bullet)$ be a complex of right R-modules, and let $(D_\bullet, d^D_\bullet)$ be a complex of left R-modules, respectively. Consider the right R-module $\mathbf{C} = \oplus_{n\in\mathbb{Z}} C_n$ and R-module endomorphism $\mathbf{d} : \mathbf{C} \to \mathbf{C}$ given by $\mathbf{d}(c_n) = d_n(c_n)$ for all $n \in Z$. In this way, we may view the differential on $C_\bullet$ as a graded homomorphism of degree -1. This is used in combination with the sign convention (7.1) next.

Let $B_{i,j} = C_i \otimes_R D_j$, a $\mathbb{Z}$-module for each i, j, and consider the following maps on $B_{i,j}$:

$$d^h_{i,j}(x \otimes_R y) = d^C_i(x) \otimes_R y \quad \text{ and } \quad d^v_{i,j}(x \otimes_R y) = (-1)^i x \otimes_R d^D_j(y)$$

for all $x \in C_i$, $y \in D_j$. That is, by using the sign convention (7.1), we may write

$$d^h_{i,j} = d^C_i \otimes_R 1_D \quad \text{ and } \quad d^v_{i,j} = 1_C \otimes_R d^D_j.$$

It may be checked that $B_{\bullet,\bullet}$ is a double complex of $\mathbb{Z}$-modules. (If R is commutative, then B is a double complex of R-modules.) The double complex $B_{\bullet,\bullet}$ is often denoted by $C_\bullet \otimes_R D_\bullet$ or $C \otimes_R D$.

Künneth Theorem and proof. The following theorem relates the homologies of two complexes of R-modules to the homology of their tensor product complex.

Theorem 7.2 (Künneth Theorem). *Let $(C_\bullet, d^C_\bullet)$ and $(D_\bullet, d^D_\bullet)$ be complexes of right and left R-modules, respectively, such that for each $n \in \mathbb{Z}$, the right R-modules C_n and $d^C_n(C_n)$ are flat. Then for each $n \in \mathbb{Z}$, there is a short exact sequence of abelian groups:*

$$0 \longrightarrow \bigoplus_{i+j=n} \mathrm{H}_i(C) \otimes_R \mathrm{H}_j(D) \longrightarrow \mathrm{H}_n(C \otimes_R D) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}^R_1(\mathrm{H}_i(C), \mathrm{H}_j(D)) \longrightarrow 0.$$

Remark 7.3. A careful check of the proof of the theorem shows that the flatness hypothesis on C_n and $d^C_n(C_n)$ can be replaced by the hypothesis that D_n and $d^D_n(D_n)$ are flat as left R-modules for each n.

For the proof, we will use some exact sequences built from cycles and boundaries of C and D as follows. Let $Z_\bullet^C = \{Z_i(C) \mid i \in \mathbb{Z}\}$ and $B_\bullet^C = \{B_i(C) \mid i \in \mathbb{Z}\}$ be the sets of cycles and boundaries of C (and similarly $Z_\bullet^D$, $B_\bullet^D$ of D). Consider $Z_\bullet^C$ and $B_\bullet^C$ to be complexes in which all differentials are the zero maps. Just to maintain notational clarity in the proof here, we define notation for a degree shift in the complex B: Let

$$B'_i = B^C_{i-1}$$

for all i, and again consider $B'_\bullet$ to be a complex in which all differentials are the zero maps. Note that the differentials on C give a chain map $d_\bullet^C : C_\bullet \to B'_\bullet$. This is then part of an exact sequence of chain complexes in which ι denotes inclusion:

$$0 \to Z_\bullet^C \xrightarrow{\iota} C_\bullet \xrightarrow{d^C} B'_\bullet \to 0. \tag{7.4}$$

Before getting to the proof of the Künneth Theorem, we state and prove a lemma.

Lemma 7.5. *Under the hypotheses of Theorem 7.2,* $\operatorname{Ker}(d_n^C)$ *is also flat for all* n.

Proof. We will work with the above exact sequence (7.4) of chain complexes of right R-modules. Consider for each n the corresponding short exact sequence of right R-modules,

$$0 \to Z_n^C \to C_n \to B'_n \to 0.$$

Apply the first long exact sequence for Tor, for any left R-module M:

$$\cdots \to \operatorname{Tor}_2^R(B'_n, M) \to \operatorname{Tor}_1^R(Z_n^C, M) \to \operatorname{Tor}_1^R(C_n, M) \to \operatorname{Tor}_1^R(B'_n, M) \to \cdots$$

Now, by the flatness hypothesis of the theorem, C_n and B'_n are flat, and so all terms above are 0 (see Remark 3.3) except possibly $\operatorname{Tor}_1^R(Z_n, M)$. However, by exactness of the sequence, that term is also now forced to be 0. It follows that Z_n^C is flat (recall Remark 4.8). □

Proof of Theorem 7.2. First, it can be checked that inclusion of cycles induces a well-defined homomorphism of abelian groups,

$$\zeta : \bigoplus_{i+j=n} \mathrm{H}_i(C) \otimes_R \mathrm{H}_j(D) \to \mathrm{H}_n(C \otimes_R D).$$

This will be the leftmost nonzero map in the sequence of the theorem. We will show that this map ζ is injective, and determine its cokernel.

For each i, j, consider the first long exact sequence for Tor as applied to D_j and $0 \to Z_i^C \to C_i \to B'_i \to 0$:

$$\cdots \to \operatorname{Tor}_1^R(B'_i, D_j) \to Z_i^C \otimes_R D_j \to C_i \otimes_R D_j \to B'_i \otimes_R D_j \to 0.$$

By hypothesis, B'_i is flat, and so $\operatorname{Tor}_1^R(B'_i, D_j) = 0$ (see Remark 3.3). Therefore as part of the above long exact Tor sequence, there is a short exact sequence for each i, j:

$$0 \to Z_i^C \otimes_R D_j \to C_i \otimes_R D_j \to B'_i \otimes_R D_j \to 0.$$

Now for each n, sum over i, j with $i + j = n$ to obtain a short exact sequence of chain complexes (i.e. total complexes) of abelian groups:

$$0 \to (Z^C \otimes_R D)_\bullet \to (C \otimes_R D)_\bullet \to (B' \otimes_R D)_\bullet \to 0.$$

Apply Theorem 4.2 to this short exact sequence of chain complexes to obtain a long exact sequence of homology:
(7.6)

$$\cdots \to \mathrm{H}_{n+1}(B' \otimes_R D) \xrightarrow{\partial} \mathrm{H}_n(Z^C \otimes_R D) \xrightarrow{(\iota\otimes 1)_*} \mathrm{H}_n(C \otimes_R D) \xrightarrow{(d^C\otimes 1)_*} \mathrm{H}_n(B' \otimes_R D) \xrightarrow{\partial} \cdots$$

We will analyze some parts of the above sequence (7.6). First note that since the differential on B' is the zero map, the differential on $B' \otimes_R D$ (i.e. the sum of the corresponding horizontal and vertical differentials for the double complex) is simply $1 \otimes d^D$. Next note that since B'_i is flat for each i by hypothesis, the functor $B'_i \otimes_R -$ takes injective R-module homomorphisms to injective R-module homomorphisms. Consequently, it takes kernels to kernels, and so (note the degree shift at the end, since $B'_i = B^C_{i-1}$):

$$\mathrm{Ker}(1 \otimes d^D)_n = (B' \otimes_R \mathrm{Ker}(d^D))_n = (B' \otimes_R Z^D)_n = (B^C \otimes_R Z^D)_{n-1}.$$

Also note that

$$\mathrm{Im}((1 \otimes d^D)_{n+1}) = (B' \otimes_R \mathrm{Im}(d^D))_n = (B' \otimes_R B^D)_n = (B^C \otimes_R B^D)_{n-1}.$$

As a consequence,

$$\mathrm{H}_n(B' \otimes_R D) \cong (B' \otimes_R \mathrm{H}(D))_n = (B^C \otimes_R \mathrm{H}(D))_{n-1}.$$

Now make the same argument for Z^C: By Lemma 7.5, Z^C_i is also flat for each i, and so we have similarly

$$\mathrm{H}_n(Z^C \otimes_R D) \cong (Z^C \otimes_R \mathrm{H}(D))_n.$$

Rewrite sequence (7.6) above, using these two replacements (recall the degree shift):
(7.7)

$$\cdots \to (B^C \otimes_R \mathrm{H}(D))_n \xrightarrow{\partial} (Z^C \otimes_R \mathrm{H}(D))_n \xrightarrow{(\iota\otimes 1)_*} \mathrm{H}_n(C \otimes_R D) \xrightarrow{(d^C\otimes 1)_*} (B^C \otimes_R \mathrm{H}(D))_{n-1} \xrightarrow{\partial} \cdots$$

Now we will analyze some parts of the sequence (7.7). Consider the connecting homomorphism ∂. A careful check back to the general construction of connecting homomorphisms shows that ∂ is in fact induced by the inclusion map, in each degree, of B^C into Z^C. A further check shows that, as a consequence, the map ζ mentioned at the start of the proof, when composed with the quotient map

$$Z^C \otimes_R H(D) \to \mathrm{H}(C) \otimes_R \mathrm{H}(D),$$

is precisely the map $(\iota \otimes 1)_*$ in (7.7). It follows that ζ is injective. We will next determine the cokernel of ζ.

For each i, j, pair the short exact sequence $0 \to B_i \to Z^C_i \to \mathrm{H}_i(C) \to 0$ with $\mathrm{H}_j(D)$ in the first long exact sequence for Tor to obtain (since Z^C_i is flat)
(7.8)

$$0 \to \mathrm{Tor}_1^R(\mathrm{H}_i(C), \mathrm{H}_j(D)) \to B^C_i \otimes_R \mathrm{H}_j(D) \xrightarrow{\partial} Z^C_i \otimes_R \mathrm{H}_j(D) \to \mathrm{H}_i(C) \otimes_R \mathrm{H}_j(D) \to 0.$$

Comparing with (7.7), note that by construction the maps ∂ in these two sequences indeed agree (upon summing over all $i + j = n$ in (7.8)), even though in (7.8), ∂ is not the connecting homomorphism.

We will combine the two sequences (7.7) and (7.8) to finish the proof. Note that in view of (7.7), the quotient

$$\mathrm{H}_n(C \otimes_R D)/\,\mathrm{Im}((\iota \otimes 1)_*)$$

embeds into $(B^C \otimes_R \mathrm{H}(D))_{n-1}$. By the above discussion, we may identify that quotient with $\mathrm{H}_n(C \otimes_R D)/\,\mathrm{Im}(\zeta)$. Considering (7.7), the image of $\mathrm{H}_n(C \otimes_R D)/\,\mathrm{Im}(\zeta)$

in $(B^C \otimes_R \mathrm{H}(D))_{n-1}$ under $(d^C \otimes 1)_*$ is precisely the kernel of ∂. In turn, considering (7.8), the kernel of ∂ is $\mathrm{Tor}_1^R(\mathrm{H}_i(C), \mathrm{H}_j(D))$ for each i, j, and summing over $i + j = n$ completes the proof. □

Koszul complexes. As an application of Theorem 7.2, we prove Theorem 7.10 (below) that gives conditions under which certain complexes, called Koszul complexes, are in fact projective resolutions. We define these complexes next, and then state the theorem.

Let x be a central element of a ring R. The corresponding *Koszul complex* is

$$K(x): \qquad 0 \longrightarrow R \xrightarrow{x\cdot} R \longrightarrow 0.$$

Now let $x_1, \ldots, x_n$ be central elements in R. Write $\mathbf{x} = (x_1, \ldots, x_n)$. The corresponding *Koszul complex* is the total complex (defined recursively when $n > 2$)

$$K(\mathbf{x}): \qquad \mathrm{Tot}(K(x_1) \otimes_R \cdots \otimes_R K(x_n)). \tag{7.9}$$

Note that if x is not a zero divisor of R, then $\mathrm{H}_1(K(x)) = 0$ and $\mathrm{H}_0(K(x)) \cong R/(x)$. It follows that $K(x)$ is a (free) resolution of the R-module $R/(x)$. We will make a more general statement about the Koszul complex $K(\mathbf{x})$ corresponding to the sequence $\mathbf{x}$, where the hypothesis that x not be a zero divisor is generalized to the hypothesis that $\mathbf{x}$ is a regular sequence, defined next.

Let M be a left R-module. A nonzero element $x \in R$ is a *zero divisor* of M if $xm = 0$ for some $m \neq 0$. A sequence $(x_1, \ldots, x_n)$ of central elements in R is called a *regular sequence* if x_1 is not a zero divisor of R and for each i, x_i is not a zero divisor of the R-module $R/(x_1, \ldots, x_{i-1})$.[3]

Theorem 7.10. *If $\mathbf{x} = (x_1, \ldots, x_n)$ is a regular sequence in R, then (7.9) is a free resolution of the R-module $R/(x_1, \ldots, x_n)$.*

A yet more general version of Theorem 7.10 is given in [7, Corollary 4.5.4].

By viewing a module itself as a complex concentrated in degree 0, with differentials all 0, we have the following corollary of Theorem 7.2.

Theorem 7.11 (Universal Coefficients Theorem)**.** *Let C be a complex of right R-modules for which C_n and $d(C_n)$ are flat for each n, and let M be a left R-module. There is a short exact sequence of abelian groups:*

$$0 \longrightarrow \mathrm{H}_n(C) \otimes_R M \longrightarrow \mathrm{H}_n(C \otimes_R M) \longrightarrow \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C), M) \longrightarrow 0.$$

Hom complexes. Let $(C_\bullet, d_\bullet^C)$ and $(D_\bullet, d_\bullet^D)$ be complexes of left (or right) R-modules. Let $B_{i,j} = \mathrm{Hom}_R(C_i, D_j)$, a $\mathbb{Z}$-module for each i, j, and consider the following maps on $B_{i,j}$:

$$d^h_{i,j}(f) = (-1)^{i-j+1} f d^C_{i+1} \quad \text{and} \quad d^v_{i,j}(f) = d^D_j f$$

for all $f \in \mathrm{Hom}_R(C_i, D_j)$. Then $B_{\bullet,\bullet}$ is a double complex of $\mathbb{Z}$-modules.

Caution: There are other choices of sign conventions on Hom complexes in the literature! Instead of our indexing choice here, it is common to reindex either C or D so that it becomes a cocomplex, before indexing the Hom complex. The double complex $B_{\bullet,\bullet}$ is often denoted by $\mathrm{Hom}_R(C_\bullet, D_\bullet)$ or $\mathrm{Hom}_R(C, D)$.

[3] In this last part, the notation $(x_1, \ldots, x_{i-1})$ refers to the ideal generated by these elements.

8. Spectral sequences

In this section, we will define cohomology spectral sequences. For homology spectral sequences, see e.g. [7] or other standard sources; these are similar but have differentials in reverse order from ours here.

Let $\mathcal{C}$ be an abelian category. A *cohomology spectral sequence* in $\mathcal{C}$ is a set $\{E_r^{pq} \mid p, q, r \in \mathbb{Z},\ r \geq 0\}$ consisting of objects of $\mathcal{C}$, together with morphisms

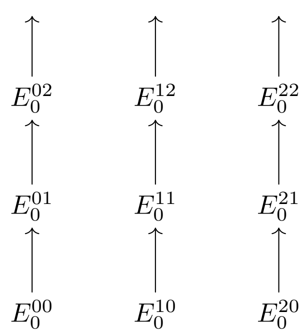
$$d_r^{pq} : E_r^{pq} \to E_r^{p+r, q-r+1}$$

satisfying $d_r^2 = 0$ (more precisely, $d_r^{p+r,q-r+1} d_r^{pq} = 0$ for all p, q, r) and $E_{r+1} \cong \mathrm{H}^{\bullet}(E_r)$ for all r. In more detail this means that for all p, q, r,

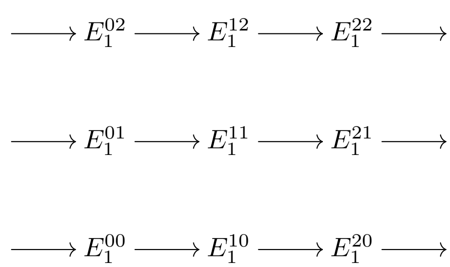
$$E_{r+1}^{p,q} \cong \mathrm{Ker}(d_r^{pq}) / \mathrm{Im}(d_r^{p-r, q+r-1}).$$

For each r, the *rth page* of the spectral sequence is the set E_r of objects E_r^{pq} together with the morphisms d_r^{pq}. For example, the pages E_0, E_1, E_2 can be given pictorially as follows.

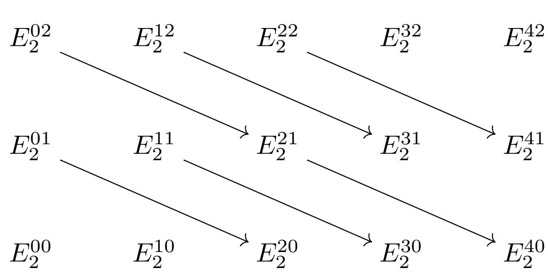
$\underline{E_0}$



$\underline{E_1}$



$\underline{E_2}$



A spectral sequence (E,d) is called *bounded* if for each integer n there are only finitely many nonzero E_0^{pq} with $p+q=n$. If (E,d) is bounded, we claim that for each fixed pair p,q, there is some integer r_0 (depending on p and q) for which $E_r^{pq} \cong E_{r_0}^{pq}$ whenever $r \geq r_0$. To see this, note that the differentials d_r^{pq} take E_r^{pq} to $E_r^{p+r,\ q-r+1}$ with fixed sum of superscripts $(p+r)+(q-r+1)=p+q+1$. Note that since E is bounded, for a fixed pair p,q, there is some r' for which $E_{r'}^{p+r',\ q-r'+1}=0$. This forces the differential with codomain $E_{r'}^{p+r',\ q-r'+1}$ (and domain $E_r^{p,q}$) to be 0. Similarly, there is some r'' for which $E_{r''}^{p-r'',\ q+r''-1}=0$, which forces the differential with domain $E_{r''}^{p-r'',\ q+r''-1}$ (and codomain $E_r^{p,q}$) to be 0. Now set $r_0=\max\{r',r''\}$. Due to the 0 differentials, the homology on page r_0 does not change (up to isomorphism) at this position p,q, that is,

$$E_{r_0+1}^{pq} \cong \mathrm{H}^\bullet(E_{r_0}^{pq}) \cong E_{r_0}^{pq}.$$

Continuing, it also follows that $E_r^{pq} \cong E_{r_0}^{pq}$ whenever $r \geq r_0$, as claimed. In this case, as a notational convention, we write

$$E_\infty^{pq} = E_{r_0}^{pq}.$$

Convergence. Let (E,d) be a bounded spectral sequence. We say that (E,d) *converges* if there exist objects H^n $(n \in \mathbb{Z})$ of $\mathcal{C}$, each having a finite filtration

$$0 = F^tH^n \hookrightarrow \cdots \hookrightarrow F^{p+1}H^n \hookrightarrow F^pH^n \hookrightarrow F^{p-1}H^n \hookrightarrow \cdots \hookrightarrow F^sH^n = H^n,$$

and isomorphisms $E_\infty^{pq} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$ for all p,q (by the notation $\hookrightarrow$ we mean monomorphisms, and by the quotient notation we mean the appropriate cokernels). We will see next how such filtrations appear naturally in some common types of spectral sequences arising from double complexes.

Spectral sequence of a double complex. We will use some standard notation for a double complex: Write $B=(B^{pq},d',d'')$ where d' and d'' are the horizontal and vertical differentials. For each n, set $B^n = \mathrm{Tot}(B)_n = \oplus_{p+q=n}B^{p,q}$, which is a complex with $d=d'+d''$, as we have seen. To refer to this complex, we will sometimes simply use the notation B.

Given a double complex B, let

$$F^pB^n = \bigoplus_{p' \geq p} B^{p',n-p'} \tag{8.1}$$

for each p and n. This can be visualized as truncating the double complex B^n at the pth column, with all objects to the left of this column replaced by 0, and then taking the direct sum of objects in the diagonal lines whose indices sum to a fixed value.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{p,2} & \longrightarrow & B^{p+1,2} & \longrightarrow & B^{p+2,2} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{p,1} & \longrightarrow & B^{p+1,1} & \longrightarrow & B^{p+2,1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
B^{p,0} & \longrightarrow & B^{p+1,0} & \longrightarrow & B^{p+2,0} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

If the double complex B is bounded, then this will be a finite filtration of B^n for each n. If B is a first quadrant double complex (that is, $B^{pq} = 0$ whenever $p < 0$ or $q < 0$), then $F^0 B^n = B^n$ and $F^p B^n = 0$ for all $p > n$.

Now let

$$C_r^{pq} = \{x \in F^p B^{p+q} \mid d(x) \in F^{p+r} B^{p+q+1}\}$$

for each p, q, r. Consequently, $C_0^{pq} = F^p B^{p+q}$. Note that if $x \in C_r^{pq}$, by the definitions, $d(x)$ is in $F^{p+r} B^{p+q+1}$, and so will have 0 component whenever $p \leq p' \leq p + r$. Set $E_0^{pq} = B_0^{pq}$ and

$$E_r^{pq} = \frac{C_r^{pq} + F^{p+1} B^{p+q}}{d(C_{r-1}^{p-r+1,q+r-2}) + F^{p+1} B^{p+q}} \tag{8.2}$$

for each $r > 0$ and $p, q \in \mathbb{Z}$. We see that there are induced morphisms

$$d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$$

for which $d_r^2 = 0$. Again it can be checked that by the definitions,

$$\mathrm{H}^\bullet(E_r) \cong E_{r+1}.$$

Remark 8.3. Often $E_1^{p,q}$ is written $\mathrm{H}''(B)^{p,q}$, that is the cohomology of B with vertical differentials only. Then E_2^{pq} is written $\mathrm{H}' \, \mathrm{H}''(B)^{p,q}$, the cohomology of $\mathrm{H}''(B)^{pq}$ with respect to differentials induced by the horizontal differentials on B.

We assume from now on that a double complex B is bounded. So for each p, q, there is an r_0 for which the differentials $d_{r_0}^{pq}$ are zero maps, as are $d_{r_0}^{p-r_0,q+r_0-1}$. Thus $E_\infty^{pq} = E_{r_0}^{pq}$.

Convergence theorem. We will next state a theorem on convergence. To that end, let $n, p, q \in \mathbb{Z}$ with $n = p + q$, and let $x \in F^p B^n$ be a cocycle for which $x \notin F^{p+1} B^n$ (we then say that x "starts" at $B^{p,q}$ since it will be a sum of elements in total degree n with nonzero component in the direct summand $B^{p,q}$ of $F^p B^n$). It follows that for all $r \geq 1$, x determines an element of E_r^{pq}, and that d_r is zero on the corresponding element of E_∞^{pq}. We have thus described a map from $F^p \, \mathrm{H}^{p+q}(B)$ (the image of $\mathrm{H}^{p+q}(F^p B)$ in $\mathrm{H}^{p+q}(B)$) to E_∞^{pq}. This map is an epimorphism since

$E_\infty^{pq} = E_{r_0}^{pq}$ for some r_0. By the definitions, its kernel is $F^{p+1}\,\mathrm{H}^{p+q}(B)$. It follows that $\mathrm{H}^\bullet(B)$ is filtered, with filtration given by $F^p\,\mathrm{H}^\bullet(B)$ and

$$F^p\,\mathrm{H}^n(B)/F^{p+1}\,\mathrm{H}^n(B) \cong E_\infty^{pq}(B)$$

for fixed $p+q=n$. We have shown that E_r converges to $\mathrm{H}^\bullet(B)$, as stated in the theorem below.

Theorem 8.4. *Let B be a bounded double complex, filtered as in (8.1). Let E be the corresponding spectral sequence, given by (8.2). Then E converges to $\mathrm{H}^\bullet(B)$.*

Notation: We often write

$$E_r \implies \mathrm{H}^*(B)$$

to indicate that E converges to $\mathrm{H}^\bullet(B)$.

Remark 8.5. We could have chosen to filter the complex by truncating rows instead of columns. This results in another spectral sequence. With this choice, the E_1 page is often denoted $\mathrm{H}'(B)$, and the E_2 page $\mathrm{H}''\,\mathrm{H}'(B)$. It can be useful to compare these two spectral sequences associated to a double complex.

9. A little group cohomology

In this section, fix a commutative ring k (usually we will take k to be a field or $k=\mathbb{Z}$). Let G be a group, and denote by kG its corresponding group ring, that is the free k-module with basis G and multiplication extended k-bilinearly from that of G. Let V be a (left) kG-module (also known as a representation of G with respect to k). The *group cohomology* of G with *coefficients* in V is defined to be

$$\mathrm{H}^n(G,V) = \mathrm{Ext}^n_{kG}(k,V)$$

where k is the *trivial* kG-module on which each group element acts as the identity.

Example: Finite cyclic group. Consider the cyclic group of order m ($m \geq 2$) generated by an element g (with multiplicative notation):

$$G = \langle g \mid g^m = 1\rangle \cong \mathbb{Z}/m\mathbb{Z}.$$

We claim that the following is a free resolution of k as a kG-module:

$$\cdots \xrightarrow{T\cdot} kG \xrightarrow{(g-1)\cdot} kG \xrightarrow{T\cdot} kG \xrightarrow{(g-1)\cdot} kG \xrightarrow{\varepsilon} k \longrightarrow 0 \tag{9.1}$$

where $T = 1+g+g^2+\cdots+g^{m-1}$ and ε is the algebra homomorphism with $\varepsilon(g)=1$. To see that this is indeed a resolution, note first that it is a complex since $g^m-1=0$ and $\varepsilon(g-1)=0$. Next, it may be checked that there exists a contracting homotopy (that is a homotopy between the identity map and 0) as follows, and as a consequence, (9.1) is exact. Define $s_{-1}: k \to kG$ by $s_{-1}(1)=1$, define s_n for n even by

$$s_n(g^j) = \begin{cases} 0, & \text{if } j=0 \\ 1+g+g^2+\cdots+g^{j-1}, & \text{if } 0<j<m, \end{cases}$$

and for n odd, $n>0$, by

$$s_n(g^j) = \begin{cases} 0, & \text{if } 0 \leq j < m-1, \\ 1, & \text{if } j = m-1. \end{cases}$$

We will find the group cohomology $\mathrm{H}^n(G,k)$ for some choices of coefficients k: Truncate (9.1), apply $\mathrm{Hom}_{kG}(-,k)$, and take homology. Since $\mathrm{Hom}_{kG}(kG,k) \cong k$, we obtain the complex (reversing order from (9.1)):

$$0 \longrightarrow k \xrightarrow{(g-1)^*} k \xrightarrow{T^*} k \xrightarrow{(g-1)^*} k \xrightarrow{T^*} \cdots \tag{9.2}$$

The induced maps $(g-1)^*$ and T^* in the complex (9.2) above may be determined from the differentials in (9.1) by working with an explicit isomorphism $\mathrm{Hom}_{kG}(kG,k) \cong k$. We find that $(g-1)^*$ is the zero map since $\varepsilon(g-1) = 0$, while

$$T^*(1) = \varepsilon(T) = m,$$

so that T^* is multiplication by m. There are several specific cases we wish to consider:

(1) If k is a field of characteristic p for which p divides m, then all differentials in the complex (9.2) are 0, and therefore

$$\mathrm{H}^n(G,k) \cong k \quad \text{for all } n \in \mathbb{N}.$$

(2) If k is a field of characteristic 0, or of characteristic not dividing m, then T^* is an isomorphism since m is invertible in k. Thus $\mathrm{H}^0(G,k) \cong k$ and $\mathrm{H}^n(G,k) = 0$ for $n > 0$. (As an alternate proof of this, we could have instead used Maschke's Theorem from representation theory: the group ring kG is semisimple in this case.)

(3) If $k = \mathbb{Z}$, we find:

$$\mathrm{H}^n(G,\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{if } n \text{ is odd} \\ \mathbb{Z}/m\mathbb{Z}, & \text{if } n > 0 \text{ and } n \text{ is even.} \end{cases}$$

Lyndon-Hochschild-Serre spectral sequence. (See e.g. [6, Section 10.7].) Let k be a field or $k = \mathbb{Z}$. Let N be a normal subgroup of a group G. Denote the corresponding quotient group by

$$\overline{G} = G/N.$$

Let V be a kG-module, also viewed as a kN-module by restriction. The following theorem shows that we may obtain information about the group cohomology $\mathrm{H}^n(G,V)$ from cohomology of the normal subgroup N and quotient group $\overline{G}$.

Theorem 9.3. *There is a spectral sequence with E_2 page*

$$E_2^{p,q} = \mathrm{H}^p\left(\overline{G}, \mathrm{H}^q(N,V)\right)$$

that converges to $\mathrm{H}^{\bullet}(G,V)$.

Sketch of proof. Let

$$\cdots \xrightarrow{\partial_2''} Q_1 \xrightarrow{\partial_1''} Q_0 \longrightarrow k \longrightarrow 0$$

be a kG-projective resolution of k, and let

$$\cdots \xrightarrow{\partial_2'} P_1 \xrightarrow{\partial_1'} P_0 \longrightarrow k \longrightarrow 0 \tag{9.4}$$

be a $k\overline{G}$-projective resolution of k.

Note that kG is free as a module over kN (under left multiplication), with basis given by a set of coset representatives. Consequently, $Q_{\bullet}$ restricts to be a projective resolution of k as a kN-module. We will apply $\mathrm{Hom}_{kN}(-,V)$ to $Q_\bullet$ to obtain a complex of abelian groups:

$$0 \longrightarrow \mathrm{Hom}_{kN}(Q_0,V) \xrightarrow{(\partial_1'')^*} \mathrm{Hom}_{kN}(Q_1,V) \xrightarrow{(\partial_2'')^*} \cdots \tag{9.5}$$

We claim that there is an action of the quotient group $\overline{G}$ on each $\mathrm{Hom}_{kN}(Q_j,V)$ so that it becomes a $k\overline{G}$-module, and that the complex (9.5) above becomes in this way a complex of $k\overline{G}$-modules. The action is defined by

$$(\overline{g}\cdot f)(x) = g(f(g^{-1}\cdot x))$$

for $\overline{g} \in \overline{G}$ (with chosen representative g in G), $x \in Q_j$, and $f \in \mathrm{Hom}_{kN}(Q_j,V)$. (It can be checked that this is a well defined action of $\overline{G}$ since f is a kN-module homomorphism.) Further, it may be checked that these actions of $\overline{G}$ commute with differentials in (9.5); recall that $Q_\bullet$ was chosen to be a resolution of kG-modules. Thus (9.5) is a complex of $k\overline{G}$-modules as claimed.

We combine the complexes (9.4) and (9.5) via Hom to make a double complex B (with some reindexing in comparison to the Hom double complex at the end of Section 7):

$$B^{p,q} = \mathrm{Hom}_{k\overline{G}}(P_p, \mathrm{Hom}_{kN}(Q_q,V)), \tag{9.6}$$

with differentials induced from those of $P_\bullet$ and of $\mathrm{Hom}_{kN}(Q_\bullet,V)$. Specifically, the differentials may be given by

$$d'(f) = (-1)^{p-q+1} f \circ \partial'_{p+1}$$

for all $f \in \mathrm{Hom}_{k\overline{G}}(P_p, \mathrm{Hom}_{kN}(Q_q,V))$, and

$$d''(f) = (\partial''_{q+1})^* \circ f,$$

that is, $(d''(f)(x))(y) = (f(x))(\partial''_{q+1}(y))$ for all $x \in P_p$, $y \in Q_{q+1}$, and $f \in \mathrm{Hom}_{k\overline{G}}(P_p, \mathrm{Hom}_{kN}(Q_q,V))$. Then, as at the end of Section 7, B is indeed a double complex.

We will consider the two spectral sequences described in Section 8 for a double complex, in relation to our double complex B given in (9.6).

Filtering by columns, there is a spectral sequence E'' for which it can be checked that

$$E_2'' \cong \mathrm{H}'(\mathrm{H}''(B)) \cong \mathrm{H}\left(\overline{G}, \mathrm{H}(N,V)\right),$$

or more precisely $(E_2'')^{p,q} \cong \mathrm{H}^p\left(\overline{G}, \mathrm{H}^q(N,V)\right)$. To see this, first note that for each p, since P_p is projective, taking $\mathrm{Hom}_{k\overline{G}}(P_p,-)$ commutes with taking homology, so

$$\mathrm{H}''(\mathrm{Hom}_{k\overline{G}}(P_p, \mathrm{Hom}_{kN}(Q_\bullet,V)) \cong \mathrm{Hom}_{k\overline{G}}(P_p, \mathrm{H}^\bullet(N,V)).$$

Now take H' of $\mathrm{Hom}_{k\overline{G}}(P_\bullet, \mathrm{H}^\bullet(N,V)$ (with differentials d'' only applied to each P_p) to obtain $\mathrm{H}^\bullet\left(\overline{G}, \mathrm{H}^\bullet(N,V)\right)$, as claimed.

Filtering by rows, there is a spectral sequence E' for which it can be checked that

$$E_2' \cong \mathrm{H}''(\mathrm{H}'(B)) \cong \mathrm{H}(G,V),$$

or more precisely,

$$(E_2')^{p,q} \cong \begin{cases} \mathrm{H}^q(G,V), & \text{if } p=0 \\ 0, & \text{if } p>0. \end{cases}$$

To see this, we first note that for each q, $\mathrm{Hom}_{kN}(Q_q, V))$ is injective as a $k\overline{G}$-module since Q_q is projective.[4] Therefore, for each q,

$$\mathrm{H}'(\mathrm{Hom}_{k\overline{G}}(P_\bullet, \mathrm{Hom}_{kN}(Q_q, V)) \cong \mathrm{Hom}_{k\overline{G}}(k, \mathrm{Hom}_{kN}(Q_q, V)) \cong \mathrm{Hom}_{kG}(Q_q, V),$$

the second isomorphism holding as the image of k under a $k\overline{G}$-module homomorphism to $\mathrm{Hom}_{kN}(Q_q, V)$ is necessarily a function invariant under each $g \in G$. Note that this is concentrated on the vertical axis (i.e. $p = 0$). Now take H'' to obtain $\mathrm{H}''(\mathrm{Hom}_{kG}(Q_\bullet, V)) \cong \mathrm{H}^n(G, V)$. That is, this second spectral sequence E' collapses at page 2 (it is concentrated on the vertical axis). We conclude that the cohomology of B is isomorphic to $\mathrm{H}^\bullet(G, V)$.

Since B is a first quadrant double complex, the first spectral sequence E'' also converges, specifically to the cohomology of B up to extensions of abelian groups. A comparison of E' and E'' concludes the proof. □

As a consequence of the theorem, $\mathrm{H}^n(G, V)$ may be described, up to extensions, by the abelian groups

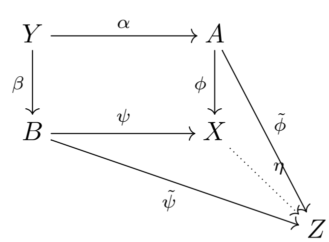
$$\left\{\mathrm{H}^p\left(\overline{G}, \mathrm{H}^q(N, V)\right) \mid p + q = n\right\}$$

together with the rest of the information contained in the first spectral sequence E''. In the nicest cases, $E''_\infty = E''_2$. We will see this in class in the example $\mathrm{H}^n(S_3, \mathbb{Z})$.

Appendix A. Supplementary definitions and constructions

Items in this appendix may not be presented in class, but may still be useful.

Pushout and pullback. Let A, B, Y be (left or right) R-modules. Let $\alpha : Y \to A$, $\beta : Y \to B$ be R-module homomorphisms. A *pushout* of α, β is an R-module X together with R-module homomorphisms $\phi : A \to X$, $\psi : B \to X$ for which the following holds: $\phi\alpha = \psi\beta$ and for any R-module Z and R-homomorphisms $\tilde{\phi} : A \to Z$, $\tilde{\psi} : B \to Z$ for which $\tilde{\phi}\alpha = \tilde{\psi}\beta$, there is a unique R-module homomorphism $\eta : X \to Z$ such that $\tilde{\phi} = \eta\phi$, $\tilde{\psi} = \eta\psi$:



This is unique up to isomorphism (by applying the definition two ways to a pair of pushouts X, X'). In fact we may take

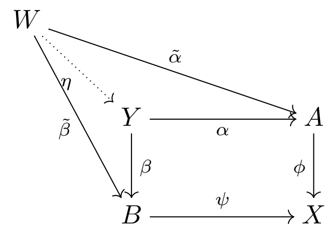
$$X = A \oplus B / \{(-\alpha(y), \beta(y)) \mid y \in Y\} \tag{A.1}$$

and ϕ, ψ the maps induced by inclusion into $A \oplus B$ followed by the quotient map.

Let A, B, X be R-modules. Let $\phi : A \to X$, $\psi : B \to X$ be R-module homomorphisms. A *pullback* of ϕ, ψ is an R-module Y together with R-module homomorphisms $\alpha : Y \to A$, $\beta : Y \to B$ for which the following holds: $\phi\alpha = \psi\beta$ and for any R-module W and R-module homomorphisms $\tilde{\alpha} : W \to A$, $\tilde{\beta} : W \to B$ for

[4] This is not obvious, and requires some representation theory to justify. (Omitted.)

which $\phi\tilde{\alpha} = \psi\tilde{\beta}$, there is a unique R-module homomorphism $\eta : W \to Y$ such that $\tilde{\alpha} = \alpha\eta$, $\tilde{\beta} = \beta\eta$.



This is unique up to isomorphism (by applying the definition two ways to a pair Y, Y' of pullbacks). In fact we may take

$$Y = \{(a,b) \in A \oplus B \mid \phi(a) = \psi(b)\}$$

and α, β the maps induced by projection from $A \oplus B$.

Extension interpretation of Ext. The abelian groups $\operatorname{Ext}^n_R(M,N)$ have an interpretation in terms of exact sequences, or extensions, as follows. An *n-extension* of M by N is an exact sequence of R-modules and R-module homomorphisms:

$$\mathbf{f}: \qquad 0 \longrightarrow N \longrightarrow U_{n-1} \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow M \longrightarrow 0.$$

Suppose $\mathbf{g}$ is another n-extension of M by N. A *morphism* of extensions from $\mathbf{f}$ to $\mathbf{g}$ is a chain map that is the identity map on each of M and N:

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & N & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U_1 & \longrightarrow & U_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_N} & & \downarrow{\scriptstyle \phi_{n-1}} & & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle 1_M} & & \\
0 & \longrightarrow & N & \longrightarrow & V_{n-1} & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & V_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Morphisms of n-extensions generate an equivalence relation. The set of equivalence classes of n-extensions becomes an abelian group under a binary operation termed the Baer sum. The n-extension $\mathbf{f}$ above has additive inverse represented by the sequence with the same modules, the map $U_0 \to M$ replaced by its additive inverse, and all other maps the same. As an abelian group, $\operatorname{Ext}^n_R(M,N)$ is isomorphic to this group of equivalence classes of n-extensions of M by N. We give the one-to-one correspondence on elements of these two groups below. We leave the proof that it is an isomorphism of abelian groups as a very lengthy exercise.

Let $\mathbf{f}$ be the n-extension of M by N given above. We wish to associate an element of $\operatorname{Ext}^n_R(M,N)$ to $\mathbf{f}$. With this goal, let $P_\bullet \to M$ be a projective resolution of M. By the Comparison Theorem (Theorem 2.7), there is a chain map $\hat{f}_\bullet$:

$$\begin{array}{ccccccccccccccc}
P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \hat{f}_n} & & \downarrow{\scriptstyle \hat{f}_{n-1}} & & & & \downarrow{\scriptstyle \hat{f}_1} & & \downarrow{\scriptstyle \hat{f}_0} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & N & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U_1 & \longrightarrow & U_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Note that by construction, $\hat{f}_n \in \operatorname{Hom}_R(P_n, N)$ and $\hat{f}_n d_{n+1} = 0$. That is, $\hat{f}_n$ is a cocycle. Write $f = \hat{f}_n$. By the Comparison Theorem again, $\hat{f}_\bullet$ is unique up to chain homotopy. It follows that the corresponding element of $\operatorname{Ext}^n_R(M,N)$ does not depend on the choice of $\hat{f}_{\bullet}$. We have thus given a map from equivalence classes of n-extensions of M by N to $\mathrm{Ext}^n_R(M,N)$.

Now let $f \in \mathrm{Hom}_R(P_n, N)$ represent an element in $\mathrm{Ext}^n_R(M,N)$, that is, $fd_{n+1} = 0$. We will construct an n-extension $\mathbf{f}$ of M by N from f. Let X be a pushout of $P_n \xrightarrow{d_n} P_{n-1}$ and $P_n \xrightarrow{f} N$. As described above, we may take

$$X = (P_{n-1} \oplus N)/\{(-d_n(x), f(x)) \mid x \in P_n\}.$$

It may be checked that the following diagram commutes:

$$\begin{array}{ccccccccccccc}
P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & P_{n-2} & \longrightarrow \cdots \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
\downarrow & & \downarrow f & & \downarrow \binom{1}{0} & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \\
0 & \longrightarrow & N & \xrightarrow{\binom{0}{1}} & X & \xrightarrow{(d_{n-1},0)} & P_{n-2} & \xrightarrow{d_{n-2}} \cdots \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0
\end{array}$$

(Equivalently, P_n may be replaced by the syzygy module $K_n = \mathrm{Ker}(d_{n-1})$ in the pushout diagram.) It may be checked that the lower sequence is an n-extension of M by N, and that any other map in $\mathrm{Hom}_R(P_n, N)$ representing the same element of $\mathrm{Ext}^n_R(M,N)$ yields an n-extension equivalent to this one. We have thus given a map from $\mathrm{Ext}^n_R(M,N)$ to the set of equivalence classes of n-extensions of M by N.

Multiplicative spectral sequences. A spectral sequence E is called *multiplicative* if there is a bigraded multiplication on E_0, that is a map

$$E_0^{p,q} \times E_0^{p',q'} \to E_0^{p+p',q+q'}$$

that satisfies the usual properties of multiplication as well as the Leibniz rule, that is,

$$d(xy) = d(x)y + (-1)^{p+q} x d(y)$$

for all $x \in E_0^{p,q}$ and $y \in E_0^{p',q'}$. It can be checked that for all r, there is an induced bigraded product on E_r for which the Leibniz rule holds. In view of Theorem 8.4, if E is multiplicative, then it converges to the associated graded algebra of $\mathrm{H}^*(B)$.

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