MATH 662 SPRING 2025
BRIEF COURSE NOTES

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These notes will include some, but not all, of the material from class. Proofs of theorems and examples are more likely to appear in class than in these notes. The appendix includes some supplementary topics that may not be covered in class.

Throughout, R will be a ring with $1 \neq 0$. Each R-module M is assumed to be *unital*, i.e. the multiplicative identity 1 of R acts as the identity map on M. We will work with both left and right modules, and where this distinction is essential, it will be specified which one.

1. Complexes

A *complex* $C_\bullet$ (or $(C_\bullet, d_\bullet)$ or (C, d)) of R-modules is a sequence of R-modules and R-module homomorphisms (called *differentials*),

$$C_\bullet : \qquad \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

for which $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The *degree* $|x|$ of an element x of C_n is n. Under this terminology, each of the differentials d_n has degree -1 as a map.

For each degree n, we define R-submodules and a subquotient of C_n as follows.

$$\begin{aligned} Z_n(C_\bullet) &= \operatorname{Ker}(d_n) && \text{(the } n\text{-cycles)} \\ B_n(C_\bullet) &= \operatorname{Im}(d_{n+1}) && \text{(the } n\text{-boundaries)} \\ H_n(C_\bullet) &= Z_n(C_\bullet)/B_n(C_\bullet) && \text{(the } n\text{th homology)} \end{aligned}$$

We say that two n-cycles x and y are *homologous* if $x - y$ is an n-boundary, that is, $x - y \in B_n$. We collect all the homology modules together and write

$$\mathrm{H}_*(C_\bullet) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_n(C_\bullet),$$

the *homology* of $C_\bullet$ (or of C, omitting the subscript for simplicity of notation). It is common to identify C with the R-module $\oplus_{n\in\mathbb{Z}} C_n$, and d with the endomorphism of this direct sum that is just d_n on each C_n as identified canonically with an R-submodule of this direct sum.

Some further terminology (that is not used universally or consistently in the literature): A *chain complex* is a complex for which $C_n = 0$ for $n < 0$. A *cochain complex* is a complex for which $C_n = 0$ for $n > 0$. These two terms are also used more generally in the literature to refer to complexes as we have defined them here.

We may wish to index complexes differently, replacing n by $-n$ in $C_\bullet$ above, with the maps still oriented as shown. Then the indexing in the above diagram is visually the same as the ordering of integers on a number line. A cochain complex then has differential of degree +1; the index is often then written as a superscript:

$$C^\bullet : \qquad 0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots$$

With this indexing, elements in the kernel of d_n are called the *n-cocycles*, and elements in the image of d_{n-1} are called the *n-coboundaries*. Two n-cocycles are called *cohomologous* if their difference is an n-coboundary. Similar to the above, we set

$$\mathrm{H}^n(C^\bullet) = \mathrm{Ker}(d_n)/\,\mathrm{Im}(d_{n-1})$$

and $\mathrm{H}^*(C^\bullet) = \oplus_{n\geq 0}\,\mathrm{H}^n(C^\bullet)$, the *cohomology* of the cochain complex $C^\bullet$.

A complex $C_\bullet$ is called *acyclic*, or *exact*, if $\mathrm{H}_n(C_\bullet) = 0$ for all n. A *short exact sequence* is an exact complex of the form $0 \to U \to V \to W \to 0$.

Let (C, d) and (C', d') be complexes. A *chain map* $f_\bullet : C_\bullet \to C'_\bullet$ is a collection of R-module homomorphisms $f_n : C_n \to C'_n$ for which $f_{n-1}d_n = d'_n f_n$ for each $n \in \mathbb{Z}$. That is, the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \\
\cdots & \longrightarrow & C'_1 & \xrightarrow{d'_1} & C'_0 & \xrightarrow{d'_0} & C'_{-1} & \longrightarrow & \cdots
\end{array}$$

It can be checked that a chain map induces a map on homology. A chain map is called a *quasi-isomorphism* if this induced map is an isomorphism on homology.

We say that two chain maps $f_\bullet, g_\bullet : C_\bullet \to C'_\bullet$ are *chain homotopic* if there exist R-module homomorphisms $s_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = s_{n-1}d_n + d'_{n+1}s_n \tag{1.1}$$

for all n. The collection $s_\bullet$ of homomorphisms is called a *homotopy* for $f_\bullet - g_\bullet$. It can be checked that chain homotopy is an equivalence relation, and that two chain homotopic maps induce the same maps on homology. As a special case, when $g_\bullet$ is the zero map, we call $s_\bullet$ a *chain contraction* of $f_\bullet$. A chain contraction of the identity map on $C_\bullet$, if it exists, is sometimes called a *contracting homotopy*, and in this case, it can be checked that $C_\bullet$ is acyclic. (In fact, for this last consequence, it is not needed that the functions s_n are R-module homomorphisms, only that there are such functions (of sets, or of abelian groups, for example) satisfying equation (1.1).)

2. Projective and injective resolutions

We call an R-module P *projective* if for every surjective R-module homomorphism $f : U \to V$ and R-module homomorphism $g : P \to V$, there exists an R-module homomorphism $h : P \to U$ such that $fh = g$:

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle g} & & \\
U & \xrightarrow{f} & V & \longrightarrow & 0
\end{array} \tag{2.1}$$

There are other equivalent definitions of projective module. For example, an R-module is projective if, and only if, it is a direct summand of a free module (i.e. $R^{\oplus I}$ for some indexing set I).

Let M be an R-module. A *projective resolution* of M is a chain complex $P_\bullet$ consisting of projective R-modules P_n $(n \geq 0)$ for which $\mathrm{H}_0(P_\bullet) \cong M$ and $\mathrm{H}_n(P_\bullet) = 0$ for all $n \neq 0$. As a consequence, $P_\bullet$ is quasi-isomorphic to the complex that is M concentrated in degree 0 and 0 elsewhere:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

Another consequence of the definition is that the following sequence is exact:

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0. \tag{2.2}$$

(In some texts, (2.2) is called a projective resolution of M.) The complex (2.2) is sometimes called the *augmented complex* of $P_\bullet$. This augmented complex may be abbreviated $P_\bullet \xrightarrow{\varepsilon} M$. The complex $P_\bullet$ (without the M) is sometimes called the *truncated complex* of (2.2).

Projective resolutions of R-modules always exist: Every R-module M is a homomorphic image of a projective R-module, for example, the free module on a set of generators of M. One may use this fact to build a projective resolution as follows. Let P_0 be a projective R-module mapping surjectively to M via an R-module homomorphism ε. Let $K_1 = \mathrm{Ker}(\varepsilon)$. In turn, K_1 is a homomorphic image of some projective R-module P_1 via some R-module homomorphism $\varepsilon_1 : P_1 \to K_1$. Denote by i_1 the inclusion map $i_1 : K_1 \to P_0$ and set $d_1 = i_1\varepsilon_1$. Let $K_2 = \mathrm{Ker}(d_1)$ and continue. Visually, we have:

$$
\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P_2 & & \xrightarrow{\quad d_2 \quad} & & P_1 & & \xrightarrow{\quad d_1 \quad} & & P_0 & \xrightarrow{\varepsilon} M \longrightarrow 0 \\
 & & & {\scriptstyle \varepsilon_2}\searrow & & \nearrow{\scriptstyle i_2} & & {\scriptstyle \varepsilon_1}\searrow & & \nearrow{\scriptstyle i_1} & & \\
 & & & & K_2 & & & & K_1 & & &
\end{array} \tag{2.3}
$$

The R-module K_i is called an *ith syzygy module* of M. This module depends on some choices. However, it is unique up to an equivalence relation, as stated in Lemma 2.5 below. We will first need Schanuel's Lemma:

Lemma 2.4 (Schanuel's Lemma)**.** *Let*

$$0 \to K \to P \xrightarrow{\varepsilon} M \to 0 \quad \textit{and} \quad 0 \to K' \to P' \xrightarrow{\varepsilon'} M \to 0$$

be two short exact sequences of R-modules with P, P' projective. Then $K \oplus P' \cong K' \oplus P$.

A proof is given in class, using the pullback of ε and ε'. For definition of pullback, see the appendix.

The next lemma is a consequence of Schanuel's Lemma via a mathematical induction argument.

Lemma 2.5. *Let K_i and K_i' be two ith syzygy modules of the R-module M. There are projective R-modules P, P' such that $K_i \oplus P \cong K_i' \oplus P'$.*

A proof is given in class.

Remark 2.6. There is another way to state Lemma 2.5: Call two R-modules U and V *equivalent* if there exist projective R-modules P, P' for which $U \oplus P \cong V \oplus P'$ as R-modules. This can be shown to be an equivalence relation. The conclusion of Lemma 2.5 is that K_i and K_i' are equivalent under this equivalence relation.

The next theorem implies a relation among projective resolutions themselves.

Theorem 2.7 (Comparison Theorem). *Let $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$ be chain complexes of R-modules with $M = \mathrm{H}_0(P_\bullet)$, $N = \mathrm{H}_0(Q_\bullet)$, and let $\varepsilon : P_0 \to M$ and $\varepsilon' : Q_0 \to N$ be corresponding augmentation maps. Assume that P_i is projective for each i and that the augmented complex $\cdots \to Q_1 \to Q_0 \to N \to 0$ is exact. If $f : M \to N$ is an R-module homomorphism, then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ for which $f\varepsilon = \varepsilon' f_0$, that is, the following diagram commutes:*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & Q_2 & \xrightarrow{d'_2} & Q_1 & \xrightarrow{d'_1} & Q_0 & \xrightarrow{\varepsilon'} & N & \longrightarrow & 0
\end{array}$$

The chain map $f_\bullet$ is unique up to chain homotopy.

Proof. A proof of existence of chain map $f_\bullet$ was given in class in the special case that both $P_\bullet$ and $Q_\bullet$ are projective resolutions. It can be checked that the "same" proof applies under these slightly more general hypotheses.

Proof of uniqueness up to chain homotopy: Let $g_\bullet : P_\bullet \to Q_\bullet$ be another chain map lifting $f : M \to N$, so that $f\varepsilon = \varepsilon' g_0$ and $d'_n g_n = g_{n-1} d_n$ for all $n \geq 1$. First note that as $\varepsilon'(f_0 - g_0) = f\varepsilon - f\varepsilon = 0$, we have

$$\mathrm{Im}(f_0 - g_0) \subseteq \mathrm{Ker}(\varepsilon') = \mathrm{Im}(d'_1).$$

Since P_0 is projective, there exists a map $s_0 : P_0 \to Q_1$:

$$\begin{array}{ccccc}
 & & P_0 & & \\
 & \swarrow{\scriptstyle s_0} & \downarrow{\scriptstyle f_0 - g_0} & & \\
Q_1 & \xrightarrow{d'_1} & \mathrm{Im}(d'_1) & \longrightarrow & 0
\end{array} \tag{2.8}$$

Setting $s_{-1} \equiv 0$, we now have $f_0 - g_0 = d'_1 s_0 = d'_1 s_0 + s_{-1}\varepsilon$.

Next we claim that $\mathrm{Im}(f_1 - g_1 - s_0 d_1) \subseteq \mathrm{Im}(d'_2)$. To see this, first compute the composition:

$$d'_1(f_1 - g_1 - s_0 d_1) = f_0 d_1 - g_0 d_1 - d'_1 s_0 d_1 = (f_0 - g_0) d_1 - d'_1 s_0 d_1 = d'_1 s_0 d_1 - d'_1 s_0 d_1 = 0.$$

It follows that $\mathrm{Im}(f_1 - g_1 - s_0 d_1) \subseteq \mathrm{Ker}(d'_1) = \mathrm{Im}(d'_2)$. Since P_1 is projective, there is consequently a map $s_1 : P_1 \to Q_2$:

$$\begin{array}{ccccc}
 & & P_1 & & \\
 & \swarrow{\scriptstyle s_1} & \downarrow{\scriptstyle f_1 - g_1 - s_0 d_1} & & \\
Q_2 & \xrightarrow{d'_2} & \mathrm{Im}(d'_2) & \longrightarrow & 0
\end{array} \tag{2.9}$$

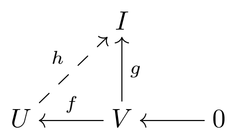
So $f_1 - g_1 - s_0 d_1 = d'_2 s_1$, that is $f_1 - g_1 = s_0 d_1 + d'_2 s_1$.

Continuing in this manner, we see that for each n, there exists a map $s_n : P_n \to Q_n$ with $f_n - g_n = s_{n-1} d_n + d'_{n+1} s_n$. That is, the two chain maps $f_\bullet$ and $g_\bullet$ are chain homotopic. $\square$

As a consequence of the Comparison Theorem, if $P_{\bullet}, Q_{\bullet}$ are projective resolutions of M, N, respectively, then there is a chain map $f_{\bullet} : P_{\bullet} \to Q_{\bullet}$ *lifting* the R-module homomorphism $f : M \to N$.

We will also work with injective resolutions. An R-module I is called *injective* if for every injective R-module homomorphism $f : V \to U$ and R-module homomorphism $g : V \to I$, there is an R-module homomorphism $h : U \to I$ for which $hf = g$:

(2.10)



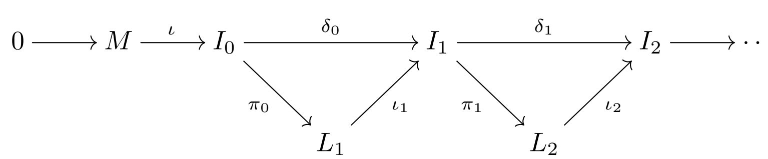
An *injective resolution* of an R-module M is a cochain complex $(I_{\bullet}, d_{\bullet})$ consisting of injective R-modules I_n $(n \leq 0)$ for which $\mathrm{H}_0(I_{\bullet}) \cong M$ and $\mathrm{H}_n(I_{\bullet}) = 0$ for all $n \neq 0$. That is, $M \cong \mathrm{Ker}(d_0)$, and the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\iota} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots, \tag{2.11}$$

where ι is an isomorphism from M to $\mathrm{Ker}(d_0)$ followed by inclusion into I_0. The complex (2.11) is sometimes called the *augmented complex* of $I_{\bullet}$. It may also itself be called an injective resolution of M. The complext $I_{\bullet}$ is sometimes called the *truncated complex* of (2.11).

Injective resolutions of R-modules always exist: Baer's Theorem states that every R-module can be embedded into an injective R-module. (See e.g. [1] for a proof of Baer's Theorem.) An injective resolution can thus be built similarly (and in reverse order) to the construction of a projective resolution above. Specifically, let $L_1 = \mathrm{Coker}(\iota) = I_0/\mathrm{Im}(\iota)$. Let $\pi_0 : I_0 \to L_1$ be the quotient map. Embed L_1 into an injective module I_1 via an R-module homomorphism $\iota_1 : L_1 \to I_1$. Let $\delta_0 = \iota_1\pi_0$. Continue by letting $L_2 = \mathrm{Coker}(\iota_1) = \mathrm{Coker}(\delta_0)$, then embed L_2 into an injective module I_2, and so on.

(2.12)



The module L_1 will be unique up to injective direct summands, due to a dual version of Schanuel's Lemma that may be checked: If $0 \to N \to I \to L \to 0$ and $0 \to N \to I' \to L' \to 0$ are exact sequences with I, I' injective, then there is an isomorphism $L \oplus I' \cong L' \oplus I$.

There is a version of Theorem 2.7 for injective resolutions. See for example [2, Comparison Theorem 2.7].

3. Ext and Tor

In this section we will define Ext and Tor.

Ext. Let M and N be R-modules. Let $P_{\bullet} \xrightarrow{\varepsilon} M$ be a projective resolution of M. Apply $\mathrm{Hom}_R(-, N)$ to the (truncated) complex $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$ to obtain a sequence of abelian groups (in reverse order for visual appeal):

$$0 \longrightarrow \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_2^*} \cdots \tag{3.1}$$

Here the abelian group homomorphism d_i^* is that induced by d_i, i.e. $d_i^*(f) = fd_i$ for all $f \in \mathrm{Hom}_R(P_{i-1}, N)$ and all $i > 0$. For convenience, we define $d_0^* = 0$.

Note that $d_{i+1}^* d_i^* = 0$ since $d_i d_{i+1} = 0$, so the above sequence (3.1) is in fact a (cochain) complex of abelian groups (that is, $\mathbb{Z}$-modules). If R is commutative, it is a complex of R-modules. If R is an algebra over a field k (that is, a ring that is also a vector space with multiplication bilinear), it is a complex of k-vector spaces. Define $\mathrm{Ext}_R^n(M, N)$ to be the cohomology of this cochain complex:

$$\mathrm{Ext}_R^n(M, N) = \mathrm{H}^n(\mathrm{Hom}_R(P_{\bullet}, N)) = \mathrm{Ker}(d_{n+1}^*)/\,\mathrm{Im}(d_n^*)$$

for $n \geq 0$, and

$$\mathrm{Ext}_R^*(M, N) = \bigoplus_{n \geq 0} \mathrm{Ext}_R^n(M, N).$$

It can be checked that, by the Comparison Theorem (Theorem 2.7), up to isomorphism of abelian groups, $\mathrm{Ext}_R^*(M, N)$ does not depend on choice of projective resolution of M. In degree 0, we have

$$\mathrm{Ext}_R^0(M, N) \cong \mathrm{Hom}_R(M, N).$$

Note that by construction, if M is itself projective, then $\mathrm{Ext}_R^n(M, N) = 0$ for all $n > 0$.

We may alternatively define $\mathrm{Ext}_R^n(M, N)$ by first taking an injective resolution of N, instead of a projective resolution of M: Take $N \xrightarrow{\iota} I_{\bullet}$ to be an injective resolution of N. Apply $\mathrm{Hom}_R(M, -)$ to the (truncated) sequence $0 \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots$ to obtain:

$$0 \longrightarrow \mathrm{Hom}_R(M, I_0) \xrightarrow{(d_0)_*} \mathrm{Hom}_R(M, I_1) \xrightarrow{(d_1)_*} \cdots \tag{3.2}$$

with $(d_i)_*(f) = d_i f$ for all i and $f \in \mathrm{Hom}_R(M, I_i)$. Set $(d_{-1})_* = 0$. It can be checked that $(d_{i+1})_*(d_i)_* = 0$ for all $i \geq -1$. Thus the sequence (3.2) is a (cochain) complex of abelian groups. It can be shown that

$$\mathrm{Ext}_R^n(M, N) \cong \mathrm{H}^n(\mathrm{Hom}_R(M, I_{\bullet})) = \mathrm{Ker}((d_n)_*)/\,\mathrm{Im}((d_{n-1})_*).$$

(A proof of the above isomorphism will at least be outlined in class.) If N is an injective R-module, it now follows that $\mathrm{Ext}_R^n(M, N) = 0$ for all $n > 0$.

Tor. Let M be a right R-module and let N be a left R-module. Let $P_{\bullet} \to M$ be a (right R-module) projective resolution of M. Apply $- \otimes_R N$ to the (truncated) complex $P_{\bullet}$ to obtain a sequence of abelian groups (i.e. $\mathbb{Z}$-modules):

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{d_2 \otimes 1_N} P_1 \otimes_R N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_R N \longrightarrow 0.$$

Here 1_N denotes the identity map on N. (In order to reduce notational clutter, we suppress the subscript R on the tensor symbol $\otimes$, just for maps and elements, when it is clear from context that the subscript on the tensor symbol $\otimes$ should be

R.) Set $d_0 = 0$. The above is a chain complex. We define $\mathrm{Tor}_n^R(M,N)$ to be its homology:

$$\mathrm{Tor}_n^R(M,N) = \mathrm{H}_n(P_\bullet \otimes_R N) = \mathrm{Ker}(d_n \otimes 1)/\,\mathrm{Im}(d_{n+1} \otimes 1)$$

for $n \geq 0$, and

$$\mathrm{Tor}_*^R(M,N) = \bigoplus_{n\geq 0} \mathrm{Tor}_n^R(M,N).$$

It can be checked that, by the Comparison Theorem (Theorem 2.7), $\mathrm{Tor}_n^R(M,N)$ does not depend on choice of projective resolution of M. It may be checked that

$$\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N.$$

We may alternatively define $\mathrm{Tor}_n^R(M,N)$ via a (left R-module) projective resolution of N: Let $Q_\bullet \to N$ be a projective resolution of N. Apply $M \otimes_R -$ to $Q_\bullet$ to obtain a sequence

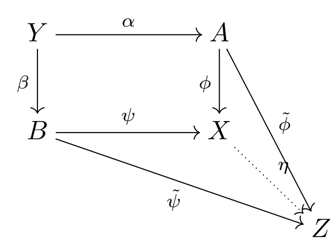
$$\cdots \longrightarrow M \otimes_R Q_2 \xrightarrow{1_M \otimes d_2} M \otimes_R Q_1 \xrightarrow{1_M \otimes d_1} M \otimes_R Q_0 \longrightarrow 0.$$

It can be proven that $\mathrm{Tor}_n^R(M,N) \cong \mathrm{H}_n(M \otimes_R Q_\bullet)$. *(Details to be discussed in class.)*

Appendix A. Supplementary definitions

Items in this appendix may not be presented in class, but may still be useful.

Pushout and pullback. Let A, B, Y be (left or right) R-modules. Let $\alpha : Y \to A$, $\beta : Y \to B$ be R-module homomorphisms. A *pushout* of α, β is an R-module X together with R-module homomorphisms $\phi : A \to X$, $\psi : B \to X$ for which the following holds: $\phi\alpha = \psi\beta$ and for any R-module Z and R-homomorphisms $\tilde{\phi} : A \to Z$, $\tilde{\psi} : B \to Z$ for which $\tilde{\phi}\alpha = \tilde{\psi}\beta$, there is a unique R-module homomorphism $\eta : X \to Z$ such that $\tilde{\phi} = \eta\phi$, $\tilde{\psi} = \eta\psi$:

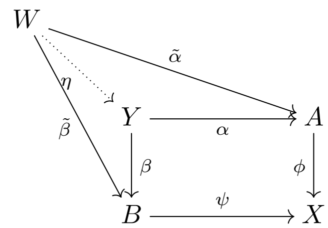


This is unique up to isomorphism (by applying the definition two ways to a pair of pushouts X, X'). In fact we may take

$$X = A \oplus B/\{(-\alpha(y), \beta(y)) \mid y \in Y\} \tag{A.1}$$

and ϕ, ψ the maps induced by inclusion into $A \oplus B$ followed by the quotient map.

Let A, B, X be R-modules. Let $\phi : A \to X$, $\psi : B \to X$ be R-module homomorphisms. A *pullback* of ϕ, ψ is an R-module Y together with R-module homomorphisms $\alpha : Y \to A$, $\beta : Y \to B$ for which the following holds: $\phi\alpha = \psi\beta$ and for any R-module W and R-module homomorphisms $\tilde{\alpha} : W \to A$, $\tilde{\beta} : W \to B$ for which $\phi\tilde{\alpha} = \psi\tilde{\beta}$, there is a unique R-module homomorphism $\eta : W \to Y$ such that $\tilde{\alpha} = \alpha\eta$, $\tilde{\beta} = \beta\eta$.



This is unique up to isomorphism (by applying the definition two ways to a pair Y, Y' of pullbacks). In fact we may take

$$Y = \{(a, b) \in A \oplus B \mid \phi(a) = \psi(b)\}$$

and α, β the maps induced by projection from $A \oplus B$.

References

[1] D. S. Passman, *A Course in Ring Theory*, Cambridge Studies in Advanced Mathematics, vol. 38, Wadsworth & Brooks/Cole, 1991.

[2] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.

[3] S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics, vol. 204, American Mathematical Society, 2019.

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