

MATH 662 SPRING 2025
BRIEF COURSE NOTES

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These notes will include some, but not all, of the material from class. Proofs of theorems and examples are more likely to appear in class than in these notes. The appendix includes some supplementary topics that may not be covered in class.

Throughout, R will be a ring with $1 \neq 0$. Each R -module M is assumed to be *unital*, i.e. the multiplicative identity 1 of R acts as the identity map on M . We will work with both left and right modules, and where this distinction is essential, it will be specified which one.

1. COMPLEXES

A *complex* C_\bullet (or (C_\bullet, d_\bullet) or (C, d)) of R -modules is a sequence of R -modules and R -module homomorphisms (called *differentials*),

$$C_\bullet : \quad \cdots \longrightarrow C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \longrightarrow \cdots$$

for which $d_{n-1}d_n = 0$ for all $n \in \mathbb{Z}$. The *degree* $|x|$ of an element x of C_n is n . Under this terminology, each of the differentials d_n has degree -1 as a map.

For each degree n , we define R -submodules and a subquotient of C_n as follows.

$$\begin{aligned} Z_n(C_\bullet) &= \text{Ker}(d_n) && \text{(the } n\text{-cycles)} \\ B_n(C_\bullet) &= \text{Im}(d_{n+1}) && \text{(the } n\text{-boundaries)} \\ H_n(C_\bullet) &= Z_n(C_\bullet)/B_n(C_\bullet) && \text{(the } n\text{th homology)} \end{aligned}$$

We say that two n -cycles x and y are *homologous* if $x - y$ is an n -boundary, that is, $x - y \in B_n$. We collect all the homology modules together and write

$$H_*(C_\bullet) = \bigoplus_{n \in \mathbb{Z}} H_n(C_\bullet),$$

the *homology* of C_\bullet (or of C , omitting the subscript for simplicity of notation). It is common to identify C with the R -module $\bigoplus_{n \in \mathbb{Z}} C_n$, and d with the endomorphism of this direct sum that is just d_n on each C_n as identified canonically with an R -submodule of this direct sum.

Some further terminology (that is not used universally or consistently in the literature): A *chain complex* is a complex for which $C_n = 0$ for $n < 0$. A *cochain complex* is a complex for which $C_n = 0$ for $n > 0$. These two terms are also used more generally in the literature to refer to complexes as we have defined them here.

We may wish to index complexes differently, replacing n by $-n$ in C_\bullet above, with the maps still oriented as shown. Then the indexing in the above diagram is

visually the same as the ordering of integers on a number line. A cochain complex then has differential of degree $+1$; the index is often then written as a superscript:

$$C^\bullet : \quad 0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \dots$$

With this indexing, elements in the kernel of d_n are called the n -cocycles, and elements in the image of d_{n-1} are called the n -coboundaries. Two n -cocycles are called *cohomologous* if their difference is an n -coboundary. Similar to the above, we set

$$H^n(C^\bullet) = \text{Ker}(d_n) / \text{Im}(d_{n-1})$$

and $H^*(C^\bullet) = \bigoplus_{n \geq 0} H^n(C^\bullet)$, the *cohomology* of the cochain complex C^\bullet .

A complex C_\bullet is called *acyclic*, or *exact*, if $H_n(C_\bullet) = 0$ for all n . A *short exact sequence* is an exact complex of the form $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$.

Let (C, d) and (C', d') be complexes. A *chain map* $f_\bullet : C_\bullet \rightarrow C'_\bullet$ is a collection of R -module homomorphisms $f_n : C_n \rightarrow C'_n$ for which $f_{n-1}d_n = d'_n f_n$ for each $n \in \mathbb{Z}$. That is, the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} & \longrightarrow & \cdots \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f_{-1} & & \\ \cdots & \longrightarrow & C'_1 & \xrightarrow{d'_1} & C'_0 & \xrightarrow{d'_0} & C'_{-1} & \longrightarrow & \cdots \end{array}$$

It can be checked that a chain map induces a map on homology. A chain map is called a *quasi-isomorphism* if this induced map is an isomorphism on homology.

We say that two chain maps $f_\bullet, g_\bullet : C_\bullet \rightarrow C'_\bullet$ are *chain homotopic* if there exist R -module homomorphisms $s_n : C_n \rightarrow C'_{n+1}$ such that

$$(1.1) \quad f_n - g_n = s_{n-1}d_n + d'_{n+1}s_n$$

for all n . The collection s_\bullet of homomorphisms is called a *homotopy* for $f_\bullet - g_\bullet$. It can be checked that chain homotopy is an equivalence relation, and that two chain homotopic maps induce the same maps on homology. As a special case, when g_\bullet is the zero map, we call s_\bullet a *chain contraction* of f_\bullet . A chain contraction of the identity map on C_\bullet , if it exists, is sometimes called a *contracting homotopy*, and in this case, it can be checked that C_\bullet is acyclic. (In fact, for this last consequence, it is not needed that the functions s_n are R -module homomorphisms, only that there are such functions (of sets, or of abelian groups, for example) satisfying equation (1.1).)

2. PROJECTIVE AND INJECTIVE RESOLUTIONS

We call an R -module P *projective* if for every surjective R -module homomorphism $f : U \rightarrow V$ and R -module homomorphism $g : P \rightarrow V$, there exists an R -module homomorphism $h : P \rightarrow U$ such that $fh = g$:

$$(2.1) \quad \begin{array}{ccc} & & P \\ & \swarrow h & \downarrow g \\ U & \xrightarrow{f} & V \longrightarrow 0 \end{array}$$

There are other equivalent definitions of projective module. For example, an R -module is projective if, and only if, it is a direct summand of a free module (i.e. $R^{\oplus I}$ for some indexing set I).

Let M be an R -module. A *projective resolution* of M is a chain complex P_\bullet consisting of projective R -modules P_n ($n \geq 0$) for which $H_0(P_\bullet) \cong M$ and $H_n(P_\bullet) = 0$ for all $n \neq 0$. As a consequence, P_\bullet is quasi-isomorphic to the complex that is M concentrated in degree 0 and 0 elsewhere:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow \varepsilon & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Another consequence of the definition is that the following sequence is exact:

$$(2.2) \quad \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

(In some texts, (2.2) is called a projective resolution of M .) The complex (2.2) is sometimes called the *augmented complex* of P_\bullet . This augmented complex may be abbreviated $P_\bullet \xrightarrow{\varepsilon} M$. The complex P_\bullet (without the M) is sometimes called the *truncated complex* of (2.2).

Projective resolutions of R -modules always exist: Every R -module M is a homomorphic image of a projective R -module, for example, the free module on a set of generators of M . One may use this fact to build a projective resolution as follows. Let P_0 be a projective R -module mapping surjectively to M via an R -module homomorphism ε . Let $K_1 = \text{Ker}(\varepsilon)$. In turn, K_1 is a homomorphic image of some projective R -module P_1 via some R -module homomorphism $\varepsilon_1 : P_1 \rightarrow K_1$. Denote by i_1 the inclusion map $i_1 : K_1 \rightarrow P_0$ and set $d_1 = i_1 \varepsilon_1$. Let $K_2 = \text{Ker}(d_1)$ and continue. Visually, we have:

$$(2.3) \quad \begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & & \searrow \varepsilon_2 & & \nearrow i_2 & \searrow \varepsilon_1 & \nearrow i_1 & & & & \\ & & & & K_2 & & K_1 & & & & \end{array}$$

The R -module K_i is called an *i th syzygy module* of M . This module depends on some choices. However, it is unique up to an equivalence relation, as stated in Lemma 2.5 below. We will first need Schanuel’s Lemma:

Lemma 2.4 (Schanuel’s Lemma). *Let*

$$0 \rightarrow K \rightarrow P \xrightarrow{\varepsilon} M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K' \rightarrow P' \xrightarrow{\varepsilon'} M \rightarrow 0$$

be two short exact sequences of R -modules with P, P' projective. Then $K \oplus P' \cong K' \oplus P$.

A proof is given in class, using the pullback of ε and ε' . For definition of pullback, see the appendix.

The next lemma is a consequence of Schanuel’s Lemma via a mathematical induction argument.

Lemma 2.5. *Let K_i and K'_i be two i th syzygy modules of the R -module M . There are projective R -modules P, P' such that $K_i \oplus P \cong K'_i \oplus P'$.*

A proof is given in class.

Remark 2.6. There is another way to state Lemma 2.5: Call two R -modules U and V *equivalent* if there exist projective R -modules P, P' for which $U \oplus P \cong V \oplus P'$ as R -modules. This can be shown to be an equivalence relation. The conclusion of Lemma 2.5 is that K_i and K'_i are equivalent under this equivalence relation.

The next theorem implies a relation among projective resolutions themselves.

Theorem 2.7 (Comparison Theorem). *Let (P_\bullet, d_\bullet) and (Q_\bullet, d'_\bullet) be chain complexes of R -modules with $M = H_0(P_\bullet)$, $N = H_0(Q_\bullet)$, and let $\varepsilon : P_0 \rightarrow M$ and $\varepsilon' : Q_0 \rightarrow N$ be corresponding augmentation maps. Assume that P_i is projective for each i and that the augmented complex $\cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow N \rightarrow 0$ is exact. If $f : M \rightarrow N$ is an R -module homomorphism, then there is a chain map $f_\bullet : P_\bullet \rightarrow Q_\bullet$ for which $f\varepsilon = \varepsilon'f_0$, that is, the following diagram commutes:*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ \cdots & \longrightarrow & Q_2 & \xrightarrow{d'_2} & Q_1 & \xrightarrow{d'_1} & Q_0 & \xrightarrow{\varepsilon'} & N & \longrightarrow & 0 \end{array}$$

The chain map f_\bullet is unique up to chain homotopy.

Proof. A proof of existence of chain map f_\bullet was given in class in the special case that both P_\bullet and Q_\bullet are projective resolutions. It can be checked that the “same” proof applies under these slightly more general hypotheses.

Proof of uniqueness up to chain homotopy: Let $g_\bullet : P_\bullet \rightarrow Q_\bullet$ be another chain map lifting $f : M \rightarrow N$, so that $f\varepsilon = \varepsilon'g_0$ and $d'_n g_n = g_{n-1} d_n$ for all $n \geq 1$. First note that as $\varepsilon'(f_0 - g_0) = f\varepsilon - \varepsilon'g_0 = 0$, we have

$$\text{Im}(f_0 - g_0) \subseteq \text{Ker}(\varepsilon') = \text{Im}(d'_1).$$

Since P_0 is projective, there exists a map $s_0 : P_0 \rightarrow Q_1$:

$$(2.8) \quad \begin{array}{ccc} & P_0 & \\ s_0 \swarrow & \downarrow f_0 - g_0 & \\ Q_1 & \xrightarrow{d'_1} & \text{Im}(d'_1) \longrightarrow 0 \end{array}$$

Setting $s_{-1} \equiv 0$, we now have $f_0 - g_0 = d'_1 s_0 = d'_1 s_0 + s_{-1} \varepsilon$.

Next we claim that $\text{Im}(f_1 - g_1 - s_0 d_1) \subseteq \text{Im}(d'_2)$. To see this, first compute the composition:

$$d'_1(f_1 - g_1 - s_0 d_1) = f_0 d_1 - g_0 d_1 - d'_1 s_0 d_1 = (f_0 - g_0) d_1 - d'_1 s_0 d_1 = d'_1 s_0 d_1 - d'_1 s_0 d_1 = 0.$$

It follows that $\text{Im}(f_1 - g_1 - s_0 d_1) \subseteq \text{Ker}(d'_1) = \text{Im}(d'_2)$. Since P_1 is projective, there is consequently a map $s_1 : P_1 \rightarrow Q_2$:

$$(2.9) \quad \begin{array}{ccc} & P_1 & \\ s_1 \swarrow & \downarrow f_1 - g_1 - s_0 d_1 & \\ Q_2 & \xrightarrow{d'_2} & \text{Im}(d'_2) \longrightarrow 0 \end{array}$$

So $f_1 - g_1 - s_0 d_1 = d'_2 s_1$, that is $f_1 - g_1 = s_0 d_1 + d'_2 s_1$.

Continuing in this manner, we see that for each n , there exists a map $s_n : P_n \rightarrow Q_n$ with $f_n - g_n = s_{n-1} d_n + d'_{n+1} s_n$. That is, the two chain maps f_\bullet and g_\bullet are chain homotopic. \square

As a consequence of the Comparison Theorem, if P_\bullet, Q_\bullet are projective resolutions of M, N , respectively, then there is a chain map $f_\bullet : P_\bullet \rightarrow Q_\bullet$ *lifting* the R -module homomorphism $f : M \rightarrow N$.

We will also work with injective resolutions. An R -module I is called *injective* if for every injective R -module homomorphism $f : V \rightarrow U$ and R -module homomorphism $g : V \rightarrow I$, there is an R -module homomorphism $h : U \rightarrow I$ for which $hf = g$:

$$(2.10) \quad \begin{array}{ccccc} & & & I & \\ & & & \uparrow g & \\ & & \nearrow h & & \\ U & \xleftarrow{f} & V & \xleftarrow{\quad} & 0 \end{array}$$

An *injective resolution* of an R -module M is a cochain complex (I_\bullet, d_\bullet) consisting of injective R -modules I_n ($n \leq 0$) for which $H_0(I_\bullet) \cong M$ and $H_n(I_\bullet) = 0$ for all $n \neq 0$. That is, $M \cong \text{Ker}(d_0)$, and the following sequence is exact:

$$(2.11) \quad 0 \longrightarrow M \xrightarrow{\iota} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots,$$

where ι is an isomorphism from M to $\text{Ker}(d_0)$ followed by inclusion into I_0 . The complex (2.11) is sometimes called the *augmented complex* of I_\bullet . It may also itself be called an injective resolution of M . The complex I_\bullet is sometimes called the *truncated complex* of (2.11).

Injective resolutions of R -modules always exist: Baer's Theorem states that every R -module can be embedded into an injective R -module. (See e.g. [1] or Hungerford for a proof of Baer's Theorem.) An injective resolution can thus be built similarly (and in reverse order) to the construction of a projective resolution above. Specifically, let $L_1 = \text{Coker}(\iota) = I_0 / \text{Im}(\iota)$. Let $\pi_0 : I_0 \rightarrow L_1$ be the quotient map. Embed L_1 into an injective module I_1 via an R -module homomorphism $\iota_1 : L_1 \rightarrow I_1$. Let $\delta_0 = \iota_1 \pi_0$. Continue by letting $L_2 = \text{Coker}(\iota_1) = \text{Coker}(\delta_0)$, then embed L_2 into an injective module I_2 , and so on.

$$(2.12) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\iota} & I_0 & \xrightarrow{\delta_0} & I_1 & \xrightarrow{\delta_1} & I_2 & \longrightarrow & \cdots \\ & & & & \searrow \pi_0 & & \nearrow \iota_1 & \searrow \pi_1 & \nearrow \iota_2 & & \\ & & & & & & L_1 & & L_2 & & \end{array}$$

The module L_1 will be unique up to injective direct summands, due to a dual version of Schanuel's Lemma that may be checked: If $0 \rightarrow N \rightarrow I \rightarrow L \rightarrow 0$ and $0 \rightarrow N \rightarrow I' \rightarrow L' \rightarrow 0$ are exact sequences with I, I' injective, then there is an isomorphism $L \oplus I' \cong L' \oplus I$.

There is a version of Theorem 2.7 for injective resolutions. See for example [2, Comparison Theorem 2.7].

3. EXT AND TOR

In this section we will define Ext and Tor.

Ext. Let M and N be R -modules. Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of M . Apply $\text{Hom}_R(-, N)$ to the (truncated) complex $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow 0$ to obtain a sequence of abelian groups (in reverse order for visual appeal):

$$(3.1) \quad 0 \longrightarrow \text{Hom}_R(P_0, N) \xrightarrow{d_1^*} \text{Hom}_R(P_1, N) \xrightarrow{d_2^*} \cdots$$

Here the abelian group homomorphism d_i^* is that induced by d_i , i.e. $d_i^*(f) = f d_i$ for all $f \in \text{Hom}_R(P_{i-1}, N)$ and all $i > 0$. For convenience, we define $d_0^* = 0$.

Note that $d_{i+1}^* d_i^* = 0$ since $d_i d_{i+1} = 0$, so the above sequence (3.1) is in fact a (cochain) complex of abelian groups (that is, \mathbb{Z} -modules). If R is commutative, it is a complex of R -modules. If R is an algebra over a field k (that is, a ring that is also a vector space with multiplication bilinear), it is a complex of k -vector spaces. Define $\text{Ext}_R^n(M, N)$ to be the cohomology of this cochain complex:

$$\text{Ext}_R^n(M, N) = H^n(\text{Hom}_R(P_\bullet, N)) = \text{Ker}(d_{n+1}^*) / \text{Im}(d_n^*)$$

for $n \geq 0$, and

$$\text{Ext}_R^*(M, N) = \bigoplus_{n \geq 0} \text{Ext}_R^n(M, N).$$

It can be checked that, by the Comparison Theorem (Theorem 2.7), up to isomorphism of abelian groups, $\text{Ext}_R^*(M, N)$ does not depend on choice of projective resolution of M . In degree 0, we have

$$\text{Ext}_R^0(M, N) \cong \text{Hom}_R(M, N).$$

Note that by construction, if M is itself projective, then $\text{Ext}_R^n(M, N) = 0$ for all $n > 0$.

We may alternatively define $\text{Ext}_R^n(M, N)$ by first taking an injective resolution of N , instead of a projective resolution of M : Take $N \xrightarrow{t} I_\bullet$ to be an injective resolution of N . Apply $\text{Hom}_R(M, -)$ to the (truncated) sequence $0 \rightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots$ to obtain:

$$(3.2) \quad 0 \longrightarrow \text{Hom}_R(M, I_0) \xrightarrow{(d_0)_*} \text{Hom}_R(M, I_1) \xrightarrow{(d_1)_*} \cdots$$

with $(d_i)_*(f) = d_i f$ for all i and $f \in \text{Hom}_R(M, I_i)$. Set $(d_{-1})_* = 0$. It can be checked that $(d_{i+1})_*(d_i)_* = 0$ for all $i \geq -1$. Thus the sequence (3.2) is a (cochain) complex of abelian groups. It can be shown that

$$\text{Ext}_R^n(M, N) \cong H^n(\text{Hom}_R(M, I_\bullet)) = \text{Ker}((d_n)_*) / \text{Im}((d_{n-1})_*).$$

(A proof of the above isomorphism will at least be outlined in class at some point.)

If N is an injective R -module, it now follows that $\text{Ext}_R^n(M, N) = 0$ for all $n > 0$.

Tor. Let M be a right R -module and let N be a left R -module. Let $P_\bullet \rightarrow M$ be a (right R -module) projective resolution of M . Apply $- \otimes_R N$ to the (truncated) complex P_\bullet to obtain a sequence of abelian groups (i.e. \mathbb{Z} -modules):

$$\cdots \longrightarrow P_2 \otimes_R N \xrightarrow{d_2 \otimes 1_N} P_1 \otimes_R N \xrightarrow{d_1 \otimes 1_N} P_0 \otimes_R N \longrightarrow 0.$$

Here 1_N denotes the identity map on N . (In order to reduce notational clutter, we suppress the subscript R on the tensor symbol \otimes , just for maps and elements, when it is clear from context that the subscript on the tensor symbol \otimes should be

R.) Set $d_0 = 0$. The above is a chain complex. We define $\text{Tor}_n^R(M, N)$ to be its homology:

$$\text{Tor}_n^R(M, N) = H_n(P_\bullet \otimes_R N) = \text{Ker}(d_n \otimes 1) / \text{Im}(d_{n+1} \otimes 1)$$

for $n \geq 0$, and

$$\text{Tor}_*^R(M, N) = \bigoplus_{n \geq 0} \text{Tor}_n^R(M, N).$$

It can be checked that, by the Comparison Theorem (Theorem 2.7), $\text{Tor}_n^R(M, N)$ does not depend on choice of projective resolution of M . It may be checked that

$$\text{Tor}_0^R(M, N) \cong M \otimes_R N.$$

We may alternatively define $\text{Tor}_n^R(M, N)$ via a (left R -module) projective resolution of N : Let $Q_\bullet \rightarrow N$ be a projective resolution of N . Apply $M \otimes_R -$ to Q_\bullet to obtain a sequence

$$\cdots \longrightarrow M \otimes_R Q_2 \xrightarrow{1_M \otimes d_2} M \otimes_R Q_1 \xrightarrow{1_M \otimes d_1} M \otimes_R Q_0 \longrightarrow 0.$$

It can be proven that $\text{Tor}_n^R(M, N) \cong H_n(M \otimes_R Q_\bullet)$. (Details to be discussed in class at some point.)

Remark 3.3. We call a left R -module F *flat* if for every short exact sequence of right R -modules $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$, the induced sequence of abelian groups $0 \rightarrow U \otimes_R F \rightarrow V \otimes_R F \rightarrow W \otimes_R F \rightarrow 0$ is exact. (There is a similar definition of flatness for right R -modules.) It may be checked that every projective module is flat. As a direct consequence of this definition, combined with the definition and balancing of Tor , if either M or N is flat as an R -module, then $\text{Tor}_n^R(M, N) = 0$ for all $n > 0$.

4. LONG EXACT SEQUENCES FOR EXT AND TOR

We first state the Snake Lemma, which has many uses. Here, we will use it to construct long exact sequences for Ext and Tor .

Lemma 4.1 (Snake Lemma). *Suppose there is a commuting diagram of R -modules and R -module homomorphisms with exact rows:*

$$\begin{array}{ccccccc} U' & \longrightarrow & V' & \xrightarrow{p} & W' & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & U & \xrightarrow{i} & V & \longrightarrow & W \end{array}$$

Then there is an exact sequence

$$\text{Ker}(f) \rightarrow \text{Ker}(g) \rightarrow \text{Ker}(h) \xrightarrow{\partial} \text{Coker}(f) \rightarrow \text{Coker}(g) \rightarrow \text{Coker}(h)$$

where $\partial(w') = i^{-1}gp^{-1}(w')$ for all $w' \in \text{Ker}(h)$.¹ If the map $U' \rightarrow V'$ is injective, then the map $\text{Ker}(f) \rightarrow \text{Ker}(g)$ is injective, and if $V \rightarrow W$ is surjective, then $\text{Coker}(g) \rightarrow \text{Coker}(h)$ is surjective.

¹Caution: This is abuse of notation. The R -module homomorphisms p, i are not assumed to be isomorphisms. Instead, by $p^{-1}(w')$ we mean any element in the inverse image of w' under p , and similarly for i^{-1} , which is then followed by canonical projection to $\text{Coker}(f)$. It can be checked that this element in $\text{Coker}(f)$ does not depend on these choices.

This lemma is often illustrated as follows, hence the name.

$$\begin{array}{ccccccc}
 & \text{Ker}(f) & \dashrightarrow & \text{Ker}(g) & \dashrightarrow & \text{Ker}(h) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & U' & \longrightarrow & V' & \xrightarrow{p} & W' & \longrightarrow 0 \\
 & \downarrow f & & \downarrow g & & \downarrow h & \\
 0 & \longrightarrow & U & \xrightarrow{i} & V & \longrightarrow & W \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Coker}(f) & \dashrightarrow & \text{Coker}(g) & \dashrightarrow & \text{Coker}(h)
 \end{array}$$

(Dashed arrows indicate commutativity and the Snake Lemma diagram.)

One consequence of the Snake Lemma (Lemma 4.1) is the following theorem that will be used to obtain long exact sequences for Tor and Ext. We define a *short exact sequence of complexes* to be a sequence

$$0 \rightarrow U. \xrightarrow{f_\bullet} V. \xrightarrow{g_\bullet} W. \rightarrow 0$$

where f_\bullet, g_\bullet are chain maps and f_i is injective, g_i is surjective, and $\text{Im}(f_i) = \text{Ker}(g_i)$ for each i .

Theorem 4.2. *Let $0 \rightarrow U. \xrightarrow{f_\bullet} V. \xrightarrow{g_\bullet} W. \rightarrow 0$ be a short exact sequence of complexes. There are abelian group homomorphisms $\partial_n : H_n(W.) \rightarrow H_{n-1}(U.)$ for each n such that*

$$\cdots \longrightarrow H_{n+1}(W) \xrightarrow{\partial_{n+1}} H_n(U) \xrightarrow{\bar{f}_n} H_n(V) \xrightarrow{\bar{g}_n} H_n(W) \xrightarrow{\partial_n} \cdots$$

is an exact sequence. (Here, \bar{f}_n, \bar{g}_n denote the maps induced by f_n, g_n .)

We call the homomorphisms ∂_n in the theorem *connecting homomorphisms*.

The next lemma will also be used in constructing long exact sequences for Ext and Tor. It is called the Horseshoe Lemma due to the shape of the given diagram.

Lemma 4.3 (Horseshoe Lemma). *Suppose there is a short exact sequence of R -modules and R -module homomorphisms $0 \rightarrow U' \rightarrow U \rightarrow U'' \rightarrow 0$. Let $P'_\bullet \rightarrow U'$, $P''_\bullet \rightarrow U''$ be projective resolutions of U' and U'' as in the diagram:*

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 \cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & U' \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & U & & \\
 & & & & \downarrow & & \\
 \cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & U'' \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
 \end{array}$$

For each n , let $P_n = P'_n \oplus P''_n$. Then there are differentials d_i for which

$$\cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \rightarrow U \rightarrow 0$$

is a projective resolution of U . Furthermore, the right column lifts to an exact sequence of complexes $0 \rightarrow P'_\bullet \xrightarrow{\iota_\bullet} P_\bullet \xrightarrow{\pi_\bullet} P''_\bullet \rightarrow 0$ for which $\iota_\bullet, \pi_\bullet$ are the canonical inclusion and projection maps.

Now we will use Theorem 4.2 and the Horseshoe Lemma (Lemma 4.3) to obtain the following four long exact sequences.

Theorem 4.4 (First long exact sequence for Ext). *Suppose U is an R -module and $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ is a short exact sequence of R -modules. Then there is an exact sequence*

$$0 \rightarrow \text{Hom}_R(U, V') \longrightarrow \text{Hom}_R(U, V) \longrightarrow \text{Hom}_R(U, V'') \longrightarrow$$

$$\text{Ext}_R^1(U, V') \longrightarrow \text{Ext}_R^1(U, V) \longrightarrow \text{Ext}_R^1(U, V'') \longrightarrow \text{Ext}_R^2(U, V') \cdots$$

Theorem 4.5 (Second long exact sequence for Ext). *Suppose V is an R -module and $0 \rightarrow U' \rightarrow U \rightarrow U'' \rightarrow 0$ is a short exact sequence of R -modules. Then there is an exact sequence*

$$0 \rightarrow \text{Hom}_R(U'', V) \longrightarrow \text{Hom}_R(U, V) \longrightarrow \text{Hom}_R(U', V) \longrightarrow$$

$$\text{Ext}_R^1(U'', V) \longrightarrow \text{Ext}_R^1(U, V) \longrightarrow \text{Ext}_R^1(U', V) \longrightarrow \text{Ext}_R^2(U'', V) \cdots$$

Theorem 4.6 (First long exact sequence for Tor). *Suppose V is a left R -module and $0 \rightarrow U' \rightarrow U \rightarrow U'' \rightarrow 0$ is an exact sequence of right R -modules. Then there is an exact sequence*

$$\cdots \longrightarrow \text{Tor}_2^R(U'', V) \longrightarrow \text{Tor}_1^R(U', V) \longrightarrow \text{Tor}_1^R(U, V) \longrightarrow$$

$$\text{Tor}_1^R(U'', V) \longrightarrow U' \otimes_R V \longrightarrow U \otimes_R V \longrightarrow U'' \otimes_R V \longrightarrow 0.$$

Theorem 4.7 (Second long exact sequence for Tor). *Suppose U is a right R -module and $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ is an exact sequence of left R -modules. Then there is an exact sequence*

$$\cdots \longrightarrow \text{Tor}_2^R(U, V'') \longrightarrow \text{Tor}_1^R(U, V') \longrightarrow \text{Tor}_1^R(U, V) \longrightarrow$$

$$\text{Tor}_1^R(U, V'') \longrightarrow U \otimes_R V' \longrightarrow U \otimes_R V \longrightarrow U \otimes_R V'' \longrightarrow 0.$$

APPENDIX A. SUPPLEMENTARY DEFINITIONS AND CONSTRUCTIONS

Items in this appendix may not be presented in class, but may still be useful.

Pushout and pullback. Let A, B, Y be (left or right) R -modules. Let $\alpha : Y \rightarrow A$, $\beta : Y \rightarrow B$ be R -module homomorphisms. A *pushout* of α, β is an R -module X together with R -module homomorphisms $\phi : A \rightarrow X$, $\psi : B \rightarrow X$ for which the following holds: $\phi\alpha = \psi\beta$ and for any R -module Z and R -homomorphisms $\tilde{\phi} : A \rightarrow Z$, $\tilde{\psi} : B \rightarrow Z$ for which $\tilde{\phi}\alpha = \tilde{\psi}\beta$, there is a unique R -module homomorphism $\eta : X \rightarrow Z$ such that $\tilde{\phi} = \eta\phi$, $\tilde{\psi} = \eta\psi$:

$$\begin{array}{ccc}
 Y & \xrightarrow{\alpha} & A \\
 \beta \downarrow & & \downarrow \phi \\
 B & \xrightarrow{\psi} & X \\
 & \searrow \tilde{\psi} & \nearrow \tilde{\phi} \\
 & & Z
 \end{array}$$

(Note: A dashed arrow η points from X to Z , and a solid arrow $\tilde{\psi}$ points from B to Z , and a solid arrow $\tilde{\phi}$ points from A to Z .)

This is unique up to isomorphism (by applying the definition two ways to a pair of pushouts X, X'). In fact we may take

$$(A.1) \quad X = A \oplus B / \{(-\alpha(y), \beta(y)) \mid y \in Y\}$$

and ϕ, ψ the maps induced by inclusion into $A \oplus B$ followed by the quotient map.

Let A, B, X be R -modules. Let $\phi : A \rightarrow X$, $\psi : B \rightarrow X$ be R -module homomorphisms. A *pullback* of ϕ, ψ is an R -module Y together with R -module homomorphisms $\alpha : Y \rightarrow A$, $\beta : Y \rightarrow B$ for which the following holds: $\phi\alpha = \psi\beta$ and for any R -module W and R -module homomorphisms $\tilde{\alpha} : W \rightarrow A$, $\tilde{\beta} : W \rightarrow B$ for which $\tilde{\phi}\tilde{\alpha} = \tilde{\psi}\tilde{\beta}$, there is a unique R -module homomorphism $\eta : W \rightarrow Y$ such that $\tilde{\alpha} = \alpha\eta$, $\tilde{\beta} = \beta\eta$.

$$\begin{array}{ccc}
 W & & \\
 \tilde{\alpha} \searrow & & \nearrow \tilde{\beta} \\
 Y & \xrightarrow{\alpha} & A \\
 \beta \downarrow & & \downarrow \phi \\
 B & \xrightarrow{\psi} & X
 \end{array}$$

(Note: A dashed arrow η points from W to Y , and a solid arrow $\tilde{\alpha}$ points from W to A , and a solid arrow $\tilde{\beta}$ points from W to B .)

This is unique up to isomorphism (by applying the definition two ways to a pair Y, Y' of pullbacks). In fact we may take

$$Y = \{(a, b) \in A \oplus B \mid \phi(a) = \psi(b)\}$$

and α, β the maps induced by projection from $A \oplus B$.

Extension interpretation of Ext. The abelian groups $\text{Ext}_R^n(M, N)$ have an interpretation in terms of exact sequences, or extensions, as follows. An *n-extension* of M by N is an exact sequence of R -modules and R -module homomorphisms:

$$\mathbf{f} : \quad 0 \longrightarrow N \longrightarrow U_{n-1} \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 \longrightarrow M \longrightarrow 0.$$

Suppose \mathbf{g} is another n -extension of M by N . A *morphism* of extensions from \mathbf{f} to \mathbf{g} is a chain map that is the identity map on each of M and N :

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U_1 & \longrightarrow & U_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1_N & & \phi_{n-1} & & & & \phi_1 & & \phi_0 & & 1_M & & \\ 0 & \longrightarrow & N & \longrightarrow & V_{n-1} & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & V_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Morphisms of n -extensions generate an equivalence relation. The set of equivalence classes of n -extensions becomes an abelian group under a binary operation termed the Baer sum. The n -extension \mathbf{f} above has additive inverse represented by the sequence with the same modules, the map $U_0 \rightarrow M$ replaced by its additive inverse, and all other maps the same. As an abelian group, $\text{Ext}_R^n(M, N)$ is isomorphic to this group of equivalence classes of n -extensions of M by N . We give the one-to-one correspondence on elements of these two groups below. We leave the proof that it is an isomorphism of abelian groups as a very lengthy exercise.

Let \mathbf{f} be the n -extension of M by N given above. We wish to associate an element of $\text{Ext}_R^n(M, N)$ to \mathbf{f} . With this goal, let $P_\bullet \rightarrow M$ be a projective resolution of M . By the Comparison Theorem (Theorem 2.7), there is a chain map \hat{f}_\bullet :

$$\begin{array}{ccccccccccccccc} P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & U_{n-1} & \longrightarrow & \cdots & \longrightarrow & U_1 & \longrightarrow & U_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Note that by construction, $\hat{f}_n \in \text{Hom}_R(P_n, N)$ and $\hat{f}_n d_{n+1} = 0$. That is, \hat{f}_n is a cocycle. Write $f = \hat{f}_n$. By the Comparison Theorem again, \hat{f}_\bullet is unique up to chain homotopy. It follows that the corresponding element of $\text{Ext}_R^n(M, N)$ does not depend on the choice of \hat{f}_\bullet . We have thus given a map from equivalence classes of n -extensions of M by N to $\text{Ext}_R^n(M, N)$.

Now let $f \in \text{Hom}_R(P_n, N)$ represent an element in $\text{Ext}_R^n(M, N)$, that is, $f d_{n+1} = 0$. We will construct an n -extension \mathbf{f} of M by N from f . Let X be a pushout of $P_n \xrightarrow{d_n} P_{n-1}$ and $P_n \xrightarrow{f} N$. As described above, we may take

$$X = (P_{n-1} \oplus N) / \{(-d_n(x), f(x)) \mid x \in P_n\}.$$

It may be checked that the following diagram commutes:

$$\begin{array}{ccccccccccccccc} P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} & X & \xrightarrow{\begin{pmatrix} 1 \\ -d_{n-1}(0) \end{pmatrix}} & P_{n-2} & \xrightarrow{d_{n-2}} & \cdots & \longrightarrow & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \end{array}$$

(Equivalently, P_n may be replaced by the syzygy module $K_n = \text{Ker}(d_{n-1})$ in the pushout diagram.) It may be checked that the lower sequence is an n -extension of M by N , and that any other map in $\text{Hom}_R(P_n, N)$ representing the same element of $\text{Ext}_R^n(M, N)$ yields an n -extension equivalent to this one. We have thus given a map from $\text{Ext}_R^n(M, N)$ to the set of equivalence classes of n -extensions of M by N .

REFERENCES

- [1] D. S. Passman, *A Course in Ring Theory*, Cambridge Studies in Advanced Mathematics, vol. 38, Wadsworth & Brooks/Cole, 1991.
- [2] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.
- [3] S. Witherspoon, *Hochschild Cohomology for Algebras*, Graduate Studies in Mathematics, vol. 204, American Mathematical Society, 2019.

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