Math 367 Notes: Mathematical Language and Reasoning 1

Teaching mathematics is a specialized kind of mathematical work.

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Mathematical knowledge needed for teaching includes precise mathematical language, reasoning, and proof.

Statements.

A *statement* is a declarative sentence that is either true or false.

Examples 1. For each of the following, determine whether it is a statement.

- It is raining in College Station right now. Yes (False)
- Is it raining? No (it's a question)
- Be quiet! No
- Owls are birds. Yes (True)
- Frogs are mammals. Yes (False)
- She was born in Louisiana. ← Depends on who "she" is
  Christy was born in Louisiana. yes. (False.)
- $2 > 1$. Yes (True)
- $1 + 1 = 3$. Yes (False)
- $x - 2 = 1$. No (depends on x)
- If $x = 3$, then $x - 2 = 1$. Yes (True.)

Quantifiers.

A *quantifier* is a word or phrase indicating quantity or amount. We will consider two types in particular: *universal quantifiers* such as "each", "all", "every", "any", "no", etc., and *existential quantifiers* such as "there exist", "there is", "there are", "some", "at least one", etc.

Examples 2. Identify the quantifier(s) in each statement:

- Every square is a rectangle. universal quantifier (True)
- There exist rectangles that are not squares. existential quantifier (True)

- For all real numbers x, $x^2 \geq 0$. universal quantifier (True)

- There is a real number x such that $x^2 < 0$. existential quantifier (False)

- No odd integer is divisible by 2. universal quantifier (True)

Negation. A *negation* of a statement is a statement that has the opposite truth value, i.e. it is true if the original statement is false, and it is false if the original statement is true.

Examples 3. Negate each statement:

- For all real numbers x, $x^2 \geq 1$. (False)
  There exist real numbers x for which $x^2 < 1$. (True: e.g. $x=0$ or $x=0.5$)
- There is a square that is not a rectangle. (False)
  All squares are rectangles. (True)
- All polygons have five sides. (False)
  There exists a polygon that does not have 5 sides. (True.)
- The sum of the angles in a triangle is 180°. (True - in Euclidean geometry.)
  every
  universal
  Negation: There is a triangle whose angles do not sum to 180°.

Proofs.
A *proof* is a demonstration that a statement is true. Euclid developed geometry in a mathematically rigorous way, including proofs based on a system of axioms. In this course, we will study a modified version of Euclid's approach. This point of view has relevance beyond mathematics. For example, Abraham Lincoln believed that mastering Euclidean geometry made him a better lawyer, and he studied Euclid's *Elements* until he could demonstrate all the statements.

Examples 4. For each statement, determine whether it is true or false. If true, prove it. If false, find a counterexample.

- The sum of two even integers is even.
- The sum of an even integer and an odd integer is odd.
- The square of an even integer is divisible by 4.
- The sum of two even integers is divisible by 4.
- The product of two odd integers is odd.