Math 367 Notes: Mathematical Language and Reasoning 2

Some statements are *compound* statements, that is, they are built from two or more simpler statements. We will look at some specific types of compound statements: conjunctions, disjunctions, implications, and biconditional statements.

Conjunctions.
Let P and Q be statements. Their *conjunction* is the statement "P and Q", true when both P and Q are true (and false when either P or Q or both are false).

Dallas and Houston are in Texas.
(P: Dallas is in Texas.
Q: Houston is in Texas.)

To prove that a conjunction is true, you must prove that both parts are true.

Disjunctions.
The *disjunction* of statements P and Q is the statement "P or Q", true when either P or Q or both are true (and false when both P and Q are false).

You may take a fitness class or exercise on your own.
(P: ...take a fitness class
Q: ... exercise on your own)

To prove that a disjunction is true, you must prove that P is true or Q is true.

Remark: To be precise, one sometimes uses the term *inclusive disjunction* to refer to a disjunction as defined above. By contrast, sometimes one might intend an *exclusive disjunction* of P and Q, which is true if either (1) P is true and Q is false or (2) P is false and Q is true (and false otherwise). In everyday language, both types of disjunctions are used, and context should make it clear which is intended. In mathematics, a disjunction is generally intended to be inclusive.

Exclusive disjunction:
You may wear your sneakers or your sandals.

Negations of conjunctions and disjunctions.
Negation interchanges these two types of compound statements.

negation of "P and Q" is "not P or not Q"

Example 1. Negate each statement:

- Ellen plays tennis and soccer.

~~E~~ Ellen does not play tennis or Ellen does not play soccer.

- James went to Brenham or to Conroe.

James did not go to Brenham and James did not go to Conroe.

Implications.
An *implication* is a statement of the form "if P, then Q". It is true if either (1) P is true and Q is true or (2) P is false. In this way, it is logically equivalent to (i.e. has the same truth value as) the disjunction "not P or Q". The *hypothesis* is the statement P, and the *conclusion* is the statement Q. We say that P is a *sufficient* condition for Q, and that Q is a *necessary* condition for P.

To prove that an implication is true, you must show that if P is true, then Q must also be true. (No need to do anything in circumstances where P is false.)

Example 2. For each statement, determine whether it is true or false. If true, prove it. If false, find a counterexample.

- For every integer n, if n is even, then n^2 is divisible by 4. True

Proof: Let n be an even integer. Then $n = 2k$ for some integer k. So $n^2 = (2k)^2 = 4k^2$, which is divisible by 4. □

- For every integer n, if n^2 is divisible by 4, then n is divisible by 4.

False. Counterexamples: 2, 6

The *converse* of the implication "if P, then Q" is "if Q, then P"; this statement is *not* logically equivalent to the original implication.

The *contrapositive* of the implication "if P, then Q" is "if not Q, then not P"; this statement *is* logically equivalent to the original implication (i.e. it has the same truth value). To prove that an implication is true, it is equivalent to prove that its contrapositive is true.

Example 3. For the following statement, write its converse and its contrapositive: If it is raining, then I take the bus.

Converse: If I take the bus, then it is raining.

Contrapositive: If I do not take the bus, then it is not raining.

Example 4. Prove the following statement by proving its contrapositive: For all integers m, if m^2 is even, then m is even.

Contrapositive: For all integers m, if m is not even, then m^2 is not even.

OR: " ", if m is odd, then m^2 is odd.

The negation of "if P, then Q" is the conjunction "P and not Q".

Biconditional statements.
A *biconditional* statement is a statement of the form "P if, and only if, Q", and this is equivalent to the conjunction "if P, then Q, and if Q, then P". To prove a biconditional statement, you must prove both implications, "if P, then Q" and "if Q, then P".

Example 5. Prove the following biconditional statement: For all integers n, n is even if, and only if, n^2 is even.