Composition and inverse.

Definition 2. Let $f : A \to B$ and $g : B \to C$ be functions. The *composition* of f and g is the function

$$g \circ f : A \to C$$

given by $(g \circ f)(a) = g(f(a))$ for all a in A.

natural numbers: $\{1, 2, 3, 4, \ldots\}$

Examples 2. Find the following compositions.

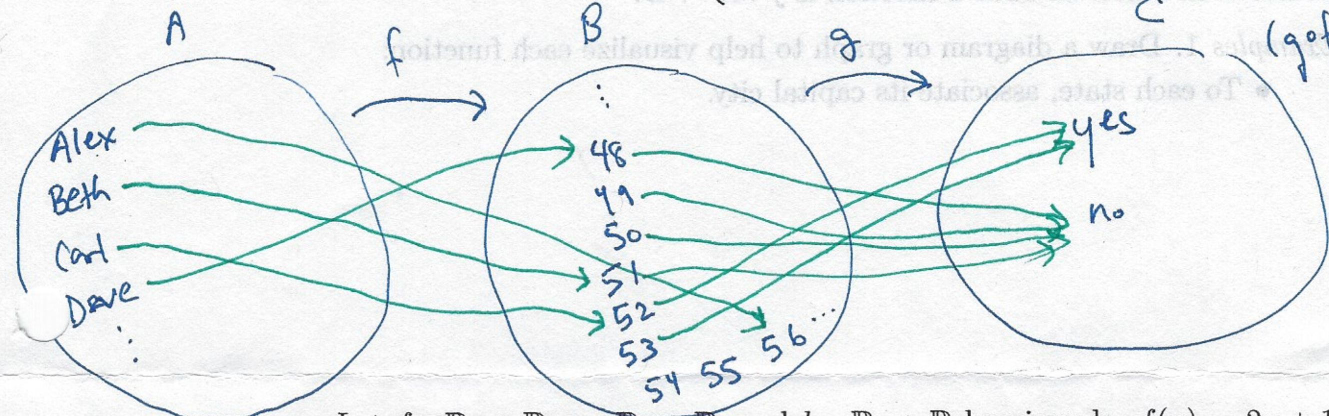
- Let A be the set of children in a fourth grade classroom. Let $B = \mathbb{N}$, and let $C = \{$ yes , no $\}$. Let $f : A \to B$ assign to each child their height in inches. Let $g : B \to C$ be given by

$$g(n) = \begin{cases} \text{yes}, & \text{if } n \geq 52 \\ \text{no}, & \text{if } n < 52. \end{cases}$$

"Bumper car function"

$(g \circ f)(\text{Alex}) = g(f(\text{Alex}))$
$= g(56) = \text{yes}$

$(g \circ f)(\text{Beth})$
$= g(51) = \text{no}$



- Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x + 1$, $g(x) = x^2$, and $h(x) = \frac{1}{2}(x - 1)$. Find the compositions $g \circ f$, $f \circ g$, $h \circ f$.

$(g \circ f)(x) = g(f(x)) = g(2x+1) = (2x+1)^2 = 4x^2 + 4x + 1$

NOT THE SAME!

$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 1$

$(h \circ f)(x) = h(f(x)) = h(2x+1) = \frac{1}{2}((2x+1) - 1) = \frac{1}{2}(2x + 1 - 1)$

$= \frac{1}{2}(2x)$

$[h(x) = \frac{1}{2}(x-1)]$

$= x$

Definition 3. Let $f : A \to B$ be a function. We say that f is *invertible* if there is a function, denoted $f^{-1} : B \to A$, such that $f^{-1} \circ f(a) = a$ for all a in A and $f \circ f^{-1}(b) = b$ for all b in B. In this case, we call f^{-1} the *inverse* function to f.

Examples 3. For each function, find its inverse if it exists. If the function is not invertible, explain why not.

- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x + 1$. [To find f^{-1}: $y = 2x+1$, switch order: $x = 2y+1$, solve for y]

Check that $f^{-1}(x) = \frac{1}{2}(x-1)$:

We already checked $(f^{-1} \circ f)(x) = x$. ✓

Check $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\frac{1}{2}(x-1)) = 2(\frac{1}{2}(x-1)) + 1 = (x-1)+1 = x$ ✓

- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sqrt[3]{x}$. [$y = \sqrt[3]{x}$, switch $x = \sqrt[3]{y}$, solve for y]

$f^{-1}(x) = x^3$

Check: $(f \circ f^{-1})(x) = f(x^3) = \sqrt[3]{x^3} = x$ ✓

$(f^{-1} \circ f)(x) = (\sqrt[3]{x})^3 = x$ ✓

- Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. *(The notation $[0, \infty)$ refers to the set of all real numbers x for which $x \geq 0$.)* $f^{-1} : \mathbb{R} \to [0, \infty)$

Check $f^{-1}(x) = x^2$

$(f \circ f^{-1})(x) = f(x^2) = \sqrt{x^2} \stackrel{?}{=} x$ $\quad (f \circ f^{-1})(-2) = f((-2)^2) = f(4) = \sqrt{4} = 2$

so $(f \circ f^{-1})(-2) \neq -2$

contradiction

NO INVERSE

- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f((x, y)) = (2x, 2y)$.

Guess $f^{-1}((x,y)) = (\frac{1}{2}x, \frac{1}{2}y)$

Check: $(f \circ f^{-1})((x,y)) = f(\frac{1}{2}x, \frac{1}{2}y) = (2(\frac{1}{2}x), 2(\frac{1}{2}y)) = (x,y)$ ✓

$(f^{-1} \circ f)((x,y)) = f^{-1}(2x, 2y) = (\frac{1}{2}(2x), \frac{1}{2}(2y)) = (x,y)$ ✓

Question. Let $f : A \to B$ and $g : B \to C$ be two invertible functions. Is $g \circ f$ invertible? If so, what is its inverse?

Yes, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, since

$(f^{-1} \circ g^{-1}) \circ (g \circ f)(x)$

$= f^{-1}(g^{-1}(g(f(x))))$

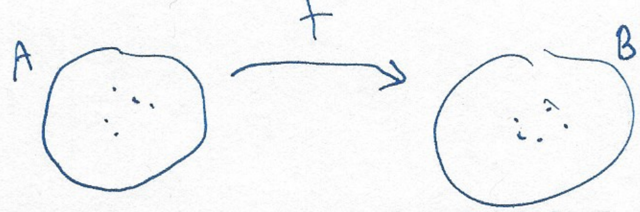
$= f^{-1}(f(x)) = x$ for all x in A

and similarly, $((g \circ f) \circ (f^{-1} \circ g^{-1}))(x) = x$ for all x in C

We worked examples on the board, e.g.

$f(x) = x^5$

$g(x) = x - 3$



Functions that are onto.

Definition 4. Let $f : A \to B$ be a function. We say f is *onto* (or *surjective*) if the following statement is true: For every element b in B, there is an element a in A for which $f(a) = b$.

Examples 4. $f : \mathbb{R} \to \mathbb{R}$

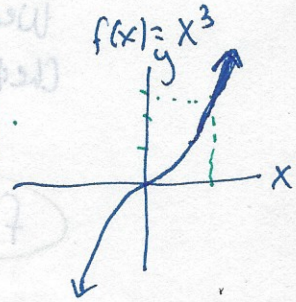
$f(x) = x^2$ NOT ONTO $\left(f(\sqrt{3}) = (\sqrt{3})^2 = 3\right)\left(f(3) = 9\right)$

because e.g. -1 is not $f(x)$ for any $x \in \mathbb{R}$

$f(x) = x^3$ ONTO: Let $y \in \mathbb{R}$. Then $y = f(x)$ for $x = \sqrt[3]{y}$.

e.g. take $-3 \in \mathbb{R}$. Then $-3 = f(\sqrt[3]{-3})$

or $-27 \in \mathbb{R}$. Then $-27 = f(-3)$



Functions that are one-to-one.

Definition 5. Let $f : A \to B$ be a function. We say f is *one-to-one* (or *injective*) if the following statement is true: For all elements a_1 and a_2 in A, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Examples 5.

$f(x) = x^2$ NOT ONE-TO-ONE: $f(2) = f(-2)$ and $2 \neq -2$

$(4 = 4)$

$f(x) = x^3$ ONE-TO-ONE: if $f(a_1) = f(a_2)$, i.e.

$a_1^3 = a_2^3$, then

$a_1 = a_2$ (since we may take cube roots)

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compare e.g. $2^3 = 2^3$ but $2^3 \neq (-2)^3$

Functions that are one-to-one correspondences.

Definition 6. Let $f : A \to B$ be a function. We say f is a *one-to-one correspondence* (or *bijective*) if f is one-to-one and onto.

Examples 6. $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$ NOT A ONE-TO-ONE CORRESPONDENCE (see previous page)

$f(x) = x^3$ ONE-TO-ONE CORRESPONDENCE (see previous page)

NOT ON EXAM

Proposition 1. *Let $f : A \to B$ be a function. Then f is invertible if, and only if, f is a one-to-one correspondence.*

Proof: ($\Rightarrow$) Assume f is invertible. Then f is onto: let $b \in B$. Set $a = f^{-1}(b)$. Then $f(a) = f(f^{-1}(b)) = b$. Also, f is one-to-one: Suppose $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$. Applying f^{-1}, we find that $f^{-1}(f(a_1)) = f^{-1}(f(a_2))$, that is, $a_1 = a_2$. Therefore f is a one-to-one correspondence.

($\Leftarrow$) Assume f is a one-to-one correspondence. Define an inverse function $f^{-1} : B \to A$ as follows: Let $b \in B$. Since f is onto, there is an element $a \in A$ such that $f(a) = b$. Set $f^{-1}(b) = a$. Since f is one-to-one, such an element a is unique, that is, there is only one element a with $f(a) = b$, so there is only one choice for $f^{-1}(b)$. Check: $f(f^{-1}(b)) = f(a) = b$ for all $b \in B$, and $f^{-1}(f(a)) = a$ for all $a \in A$, by the definition of f^{-1}. $\square$

Proposition 2. *Let $f : A \to B$ and $g : B \to C$ be functions. If f and g are both one-to-one correspondences, then $g \circ f$ is a one-to-one correspondence.*

Proof: Let f and g be one-to-one correspondences. By Proposition 1, f and g are both invertible, and by our observation on an earlier page, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, that is, $g \circ f$ is invertible.
By Proposition 1, $g \circ f$ is a one-to-one correspondence.